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Uniform Asymptotic Expansions of Solutions of Linear Second-Order Differential Equations for Large Values of a Parameter

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UNIFORM ASYMPTOTIC EXPANSIONS OF SOLUTIONS OF LINEAR SECOND-ORDER DIFFERENTIAL EQUATIONS FOR LARGE VALUES OF A PARAMETER

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CONTENTS

An investigation is made of the differential equations

$$\frac{d^2w}{dz^2} = \{u^2+f(u,z)\}w, \qquad \frac{d^2w}{dz^2} = \{u^2z+f(u,z)\}w,$$

$$\frac{d^2w}{dz^2} = \frac{1}{z}\frac{dw}{dz}+\left\{u^2+\frac{\mu^2-1}{z^2}+f(u,\mu,z)\right\}w,$$

in which u is a large complex parameter, μ is a real or complex parameter independent of u, and z is a complex variable whose domain of variation may depend on arg u and μ, and need not be bounded.

General conditions are obtained under which solutions exist having the formal series

$$w = P(z)\sum_{s=0}^{\infty}\frac{A_s}{u^{2s}}+\frac{P'(z)}{u^2}\sum_{s=0}^{\infty}\frac{B_s}{u^{2s}}$$

as their asymptotic expansions for large $|u|$, uniformly valid with respect to z, arg u and μ. Here $P(z)$ is respectively an exponential function, Airy function or Bessel function of order μ, and the coefficients A_s and B_s are given by recurrence relations.

1. Introduction and summary

In recent papers (Olver 1954*a*, 1956) I investigated the asymptotic expansion of solutions of differential equations of the form

$$\frac{d^2w}{dz^2} = \{u^2p(z)+q(z)\}w \tag{1·1}$$

for large values of the real positive parameter† u, which are uniformly valid with respect to all values of the complex variable z lying in a domain **D**, bounded or otherwise. The asymptotic character of the solutions depends on the number and nature of the *transition points* in **D**, that is, points at which $p(z)$ has a zero, or $p(z)$ or $q(z)$ has a singularity. Zeros of $p(z)$ are also called *turning points*.

Four principal cases were distinguished, denoted by A, B, C and D. In case A, **D** is free from transition points; in case B, **D** contains a simple turning point; in case C, **D** contains a double pole of $p(z)$; in case D, **D** contains a simple pole of $p(z)$. In the last two cases $q(z)$ is permitted to have a single or double pole at the pole of $p(z)$. By simultaneous change of dependent and independent variables equation (1·1) can be transformed into one of the following forms

$$\frac{d^2w}{dz^2} = \{u^2+f(z)\}w \quad \text{(cases A and C)}, \tag{1·2}$$

$$\frac{d^2w}{dz^2} = \{u^2z+f(z)\}w \quad \text{(case B)}, \tag{1·3}$$

$$\frac{d^2w}{dz^2} = \frac{1}{z}\frac{dw}{dz}+\left\{u^2+\frac{\mu^2-1}{z^2}+f(z)\right\}w \quad \text{(case D)}, \tag{1·4}$$

in which μ is a constant and $f(z)$ is a regular (holomorphic) function in the transformed domain. In (1·4) $f(z)$ is an even function of z.

Each of the equations (1·2), (1·3) and (1·4) can be satisfied formally by series of the form

$$P_j(z)\left\{1+\sum_{s=1}^{\infty}\frac{A_s(z)}{u^{2s}}\right\}+\frac{P_j'(z)}{u^2}\sum_{s=0}^{\infty}\frac{B_s(z)}{u^{2s}}, \tag{1·5}$$

in which $P_j(z)$ is a solution of the corresponding equation with $f(z) = 0$, the suffix j being associated with the particular choice of solution. The $P_j(z)$ are called the *basic functions* and are respectively exponential functions, Airy functions and Bessel functions of order μ for the three equations.

Existence theorems designated A, B and D were proved, which establish that solutions of the equations exist for which the formal series (1·5) is an asymptotic expansion, in the sense of Poincaré, for large positive u, provided that z lies in a certain subdomain $\mathbf{D}_j$ of the transformed domain corresponding to **D**. $\mathbf{D}_j$ is independent of u and can extend to infinity in regions where the condition

$$\left.\begin{aligned} f(z) &= O(|z|^{-1-\sigma}) \quad \text{(equations (1·2) and (1·4))},\\ f(z) &= O(|z|^{-\frac{1}{2}-\sigma}) \quad \text{(equation (1·3))}, \end{aligned}\right\} \tag{1·6}$$

is satisfied; σ being a positive constant.

Any other equation of the form (1·1) which has an isolated transition point and which is amenable to the same general treatment can be transformed into either (1·2), (1·3) or (1·4). In addition, Thorne (1957*a*) has shown that the case in which **D** contains both a simple turning point and a double pole of $p(z)$ can be transformed into (1·3); the pole becomes a point at infinity at which the condition (1·6) is satisfied. He has also indicated that the case in which **D** contains both a simple and a double pole of $p(z)$ can likewise be transformed into the form (1·4).

† For convenience the u of the previous papers is replaced here by u^2.

A further case, requiring a new existence theorem, has been investigated by Thorne (1957*b*), using similar methods. The standard form of equation is

$$\frac{d^2w}{dz^2} = \left\{u^2\left(1+\frac{\alpha^2}{z^2}\right) - \frac{1}{4z^2} + f(z)\right\} w, \tag{1·7}$$

where α is a constant, $f(z)$ is even and $f(z) = O(|z|^{-1-\sigma})$ at infinity. The basic functions here are Bessel functions of order αu. This equation corresponds to (1·1) when $p(z)$ has a double pole and a simple zero in **D**. The branch cut for the solutions which emanates from the pole is made to pass through the zero, and the regions of validity of the corresponding expansions include the pole and the *two* turning points, one on either side of the cut.

Although in the papers cited it is supposed that u is a real positive parameter, it is observed that if u is complex and arg u is fixed then a trivial transformation will convert any of (1·2), (1·3), (1·4) or (1·7) into the same form of equation with u replaced by $|u|$. The existence theorems accordingly apply to the situation when u is large and complex and arg u is fixed.

The principal object of the present paper is to extend theorems A, B and D by establishing conditions under which the asymptotic expansions are uniformly valid over any prescribed range of arg u. The need for this extension in cases A and B has arisen in developing an asymptotic theory for the case in which equation (1·1) contains *two* simple turning points in **D**. The basic functions for this case are the Weber parabolic cylinder functions, being the solutions of the differential equation

$$\frac{d^2w}{dz^2} = (\tfrac{1}{4}z^2 + a)\, w.$$

It is desirable to know asymptotic representations of these functions for large values of the parameter a which are uniformly valid with respect to z and arg a. Such representations can be found by application of the new forms of theorems A and B given in the present paper.

The desired extension of the existence theorems can be achieved by introducing an extra parameter θ in (1·2), (1·3) and (1·4), replacing $f(z)$ by $f(\theta, z)$. The equations become respectively

$$\frac{d^2w}{dz^2} = \{u^2 + f(\theta, z)\}\, w, \tag{1·8}$$

$$\frac{d^2w}{dz^2} = \{u^2 z + f(\theta, z)\}\, w, \tag{1·9}$$

$$\frac{d^2w}{dz^2} = \frac{1}{z}\frac{dw}{dz} + \left\{u^2 + \frac{\mu^2-1}{z^2} + f(\theta, z)\right\} w, \tag{1·10}$$

and the coefficients $A_s(z)$ and $B_s(z)$ in the series (1·5) are replaced by $A_s(\theta, z)$ and $B_s(\theta, z)$, respectively. We seek conditions under which the asymptotic property of this series *for large positive u* is uniform with respect to θ as well as z.

The connexion of this approach with the theory of (1·2), (1·3) and (1·4) for large complex u is as follows. Consider, for example, equation (1·2). If u is complex and $\theta = \arg u$, the substitutions

$$u = |u|\, e^{i\theta}, \qquad z = e^{-i\theta} z_1, \tag{1·11}$$

transform (1·2) into

$$\frac{d^2w}{dz_1^2} = \{|u|^2 + e^{-2i\theta} f(e^{-i\theta} z_1)\}\, w. \tag{1·12}$$

This equation is clearly of the form (1·8) with u positive and $f(\theta, z_1) = e^{-2i\theta} f(e^{-i\theta} z_1)$. In fact, it would be adequate for the immediate purpose if we were to replace $f(\theta, z)$ in (1·8) by $e^{-2i\theta} f(e^{-i\theta} z)$. Greater generality is achieved however, with no extra complication, by taking f to be a general function of the two variables θ and z. And although the parameter θ in (1·8), (1·9) and (1·10) corresponds to arg u in (1·2), (1·3) and (1·4) with complex u, when we consider the asymptotic theory of (1·8), (1·9) and (1·10) we merely regard θ as another parameter, real or complex, either independent of or dependent on the real positive parameter u, or even as a set of such parameters.

In establishing conditions under which the formal series (1·5), with $A_s(z)$ and $B_s(z)$ replaced by $A_s(\theta, z)$ and $B_s(\theta, z)$, represents uniform asymptotic expansions of solutions of equations (1·8), (1·9) and (1·10), it is essential that we permit the boundaries of the z-domain to depend on θ. This is evident from the transformation (1·11) which shows that the z_1-domain is obtained from the z-domain by rotation through an angle θ. We generalize this relationship and allow the z-domain $\mathbf{D}(\theta)$ associated with (1·8), (1·9) and (1·10) to be a general function of θ. This generalization is of great value in applications since it allows the boundary of $\mathbf{D}(\theta)$ to be arranged in the most convenient manner for each θ. In particular, it is usually advantageous to arrange that cuts from branch points of $f(\theta, z)$ are parallel to the imaginary z-axis in the case of (1·8) and (1·10), and lie along level curves of the function $\exp(\frac{2}{3} z^{\frac{3}{2}})$ in the case of (1·9); the regions of validity of the asymptotic expansions are then likely to be maximal.

The proofs given in this paper are simpler in principle than the proofs previously given of the original forms of the existence theorems; two of the three lemmas concerning the magnitude of the coefficients $A_s(\theta, z)$ and $B_s(\theta, z)$ are avoided. The simpler method of proof permits an easy extension to be made to meet the situation in which $f(\theta, z)$ is replaced by a function $f(u, \theta, z)$ having a uniform asymptotic expansion of the form

$$f(u, \theta, z) \sim f_0(\theta, z) + \frac{f_1(\theta, z)}{u} + \frac{f_2(\theta, z)}{u^2} + \dots, \tag{1·13}$$

for large u.

Another extension is the easing of restrictions placed upon the boundaries of the regions of validity. We no longer suppose that these consist of a finite number of straight lines. This is of particular value in case B, for here we often wish to arrange that branch cuts bounding $\mathbf{D}(\theta)$ are level curves of the function $\exp(\frac{2}{3} z^{\frac{3}{2}})$. In this case, moreover, the previous restriction that the distances of all points of the regions $\mathbf{D}_j(\theta)$ from the boundary of $\mathbf{D}(\theta)$ must have a positive lower bound, is replaced by the weaker condition: if z_0 is a boundary point of $\mathbf{D}_j(\theta)$ then $|z_0|^{\frac{1}{2}}$ times the shortest distance between z_0 and the boundary of $\mathbf{D}(\theta)$ must have a positive lower bound. This too simplifies applications as we shall see in § 10.

As our final extension, the existence theorem in case D now establishes uniformity with respect to the variable μ occurring in (1·10), when μ lies in a bounded region in the half-plane $\mathrm{Re}\,\mu \geqslant 0$. The function f and the z-domains are permitted to depend on μ as well as θ.

Uniform asymptotic solutions of second-order differential equations with a large complex parameter have been derived by many authors, though usually only for real values of the independent variable. Cherry (1950), however, develops uniform asymptotic series for case B with u complex and z lying in a given bounded star-domain. The z-regions of validity which result from these conditions are unnecessarily restricted and the greater flexibility in the choice of boundaries permitted in the present paper yields larger regions in applications, even in the case of bounded star-domains. Another condition of Cherry's paper is that the function $f(u, \theta, z)$, defined above, should have a convergent expansion in descending powers of u^2; here we merely suppose that $f(u, \theta, z)$ has an asymptotic expansion in descending powers of u. Other comparisons between the results of Cherry and those of the present writer have been discussed in Olver 1954*a*, § 7.

Kazarinoff & McKelvey (1956) have investigated an extended form of case C with both u and z complex. The z-domain is severely restricted however, being the interior of a fixed simple closed contour.

The present paper is divided into three parts, corresponding to cases A, B and D. The arrangement of each part follows the same pattern. In the opening section (§§ 2, 9 and 14) the problem is stated together with its solution, the corresponding existence theorem. The postulated conditions are of a rather general form, and to facilitate their application the next section (§§ 3, 10 and 15) is mainly devoted to an account of common circumstances in which the general conditions are fulfilled. The proof of each theorem then begins. First (§§ 5, 11 and 17 (i)) bounds are established for the coefficients of the series (1·5) which are uniform with respect to θ and, in case D, μ. Next (§§ 6, 12 and 17 (ii)), a differential equation satisfied by the truncated series is obtained. Combination of this equation with the original equation yields a differential equation for the error term of the asymptotic series, and in the final sections (§§ 7, 8, 13, 18 and 19) an integral equation corresponding to the last-named differential equation is solved by the iterative method.

In part 3 there is an additional section (§ 16) in which bounds are established for the modified Bessel functions $I_\mu(z)$ and $K_\mu(z)$ which are uniform with respect to bounded μ. These may be of use in other contexts.

PART 1. CASE A

2. Statement of conditions and theorem A

We consider the differential equation

$$\frac{d^2w}{dz^2} = \{u^2 + f(u, \theta, z)\}\, w, \tag{2·1}$$

in which u is a large positive parameter; θ is a set of real or complex parameters ranging over a set of values $\mathbf{\Theta}$; z is a complex variable ranging over a simply-connected complex domain $\mathbf{D}(\theta)$, bounded or otherwise.

We suppose that for given u and θ, $f(u, \theta, z)$ is a regular (holomorphic) function of z in $\mathbf{D}(\theta)$, and that

$$\left| f(u, \theta, z) - \sum_{s=0}^{m-1} \frac{f_s(\theta, z)}{u^s} \right| < \frac{1}{1 + |z|^{1+\sigma}} \frac{k_m}{u^m} \quad \text{(for every } m = 0, 1, 2, \ldots), \tag{2·2}$$

when $z \in \mathbf{D}(\theta)$; $\theta \in \mathbf{\Theta}$; $u \geqslant u_0 (>0)$, independent of z and θ. Each of the coefficients $f_s(\theta, z)$ is independent of u† and a regular function of z in $\mathbf{D}(\theta)$, and σ is a positive constant independent of u, θ, z and m. The number k_m is independent of u, θ and z, but may depend on m; we shall

† Except in as much as θ may depend on u.

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use the symbol k_m generically in this sense, and also the symbol k generically to denote a positive number independent of u, θ and z.

The condition (2·2) implies that

$$f(u,\theta,z) \sim \sum_{s=0}^{\infty} \frac{f_s(\theta,z)}{u^s} \tag{2·3}$$

as $u \to +\infty$, uniformly valid with respect to θ and z. We can construct a formal solution of (2·1) in the form

$$w = e^{uz} \sum_{s=0}^{\infty} \frac{A_s(\theta,z)}{u^s}. \tag{2·4}$$

Term-by-term differentiation yields

$$\frac{dw}{dz} = u\, e^{uz} \sum_{s=0}^{\infty} \frac{B_s(\theta,z)}{u^s}, \tag{2·5}$$

$$\frac{d^2w}{dz^2} = u^2 e^{uz} \sum_{s=0}^{\infty} \frac{C_s(\theta,z)}{u^s}, \tag{2·6}$$

where $$B_s = A_s + A'_{s-1}, \qquad C_s = B_s + B'_{s-1} = A_s + 2A'_{s-1} + A''_{s-2}, \tag{2·7}$$

primes denoting differentiations with respect to z. Substituting (2·3), (2·4) and (2·6) in (2·1) and equating coefficients, we see that the last-named equation is formally satisfied if

$$C_{s+2} = A_{s+2} + f_0 A_s + f_1 A_{s-1} + \ldots + f_s A_0, \tag{2·8}$$

that is, if $$2A'_{s+1} = -A''_s + f_0 A_s + f_1 A_{s-1} + \ldots + f_s A_0. \tag{2·9}$$

On integration, this yields $A_0 =$ constant, and

$$A_{s+1} = -\tfrac{1}{2}A'_s + \tfrac{1}{2}\int (f_0 A_s + f_1 A_{s-1} + \ldots + f_s A_0)\, dz \quad (s \geqslant 0). \tag{2·10}$$

This determines $A_0, A_1, A_2, \ldots$, apart from arbitrary constants of integration, which may depend on θ. Each A_s is a regular function of z in $\mathbf{D}(\theta)$, and single-valued since $\mathbf{D}(\theta)$ is simply-connected.

We can easily verify that a second formal solution of (2·1) is given by

$$w = e^{-uz} \sum_{s=0}^{\infty} (-)^s \frac{A^*_s(\theta,z)}{u^s}, \tag{2·11}$$

$$\frac{dw}{dz} = -u\, e^{-uz} \sum_{s=0}^{\infty} (-)^s \frac{B^*_s(\theta,z)}{u^s}, \tag{2·12}$$

where the coefficients are determined by $A^*_0 =$ constant,

$$A^*_{s+1} = -\tfrac{1}{2}A^{*\prime}_s + \tfrac{1}{2}\int (f_0 A^*_s - f_1 A^*_{s-1} + \ldots + (-)^s f_s A^*_0)\, dz \quad (s \geqslant 0), \tag{2·13}$$

and $$B^*_s = A^*_s + A^{*\prime}_{s-1}. \tag{2·14}$$

If the functions $f_s(\theta,z)$ of odd suffix s all vanish, then A^*_s and B^*_s are the same as A_s and B_s, respectively; in particular, this happens when $f(u,\theta,z)$ is independent of u.

We define $\mathbf{G}(\theta)$ to be an arbitrary closed subdomain of $\mathbf{D}(\theta)$, subject to the following conditions.

(i) The distance between each point of $\mathbf{G}(\theta)$ and each boundary point of $\mathbf{D}(\theta)$ has a positive lower bound which is independent of θ.

(ii) For each θ, a point $c(\theta)$ and a path joining $c(\theta)$ and z lying wholly in $\mathbf{G}(\theta)$ can be found such that

$$\int_{c(\theta)}^{z} \frac{|\,dt\,|}{1+|\,t\,|^{1+\sigma}} < k \quad (z \in \mathbf{G}(\theta)). \tag{2·15}$$

The only restriction we impose on the arbitrary constants associated with the determination of the coefficients $A_s(\theta, z)$ and $A_s^*(\theta, z)$ is that $A_s\{\theta, c(\theta)\}$ and $A_s^*\{\theta, c(\theta)\}$ are bounded functions of θ; thus

$$|\,A_s\{\theta, c(\theta)\}\,| < k_s, \qquad |\,A_s^*\{\theta, c(\theta)\}\,| < k_s. \tag{2·16}$$

In these circumstances, the asymptotic nature of the formal series may be expressed in the following theorem:

THEOREM A. *The differential equation* (2·1) *possesses solutions* $W_1(u, \theta, z)$ *and* $W_2(u, \theta, z)$ *with the properties*

$$\left.\begin{aligned} W_1(u,\theta,z) &= e^{uz}\left[\sum_{s=0}^{m} \frac{A_s(\theta,z)}{u^s} + O\left(\frac{1}{u^{m+1}}\right)\right] \\ \frac{d}{dz}W_1(u,\theta,z) &= u\,e^{uz}\left[\sum_{s=0}^{m} \frac{B_s(\theta,z)}{u^s} + O\left(\frac{1}{u^{m+1}}\right)\right] \end{aligned}\right\} \quad (z \in \mathbf{H}_1(\theta), \theta \in \mathbf{\Theta}), \tag{2·17}$$

$$\left.\begin{aligned} W_2(u,\theta,z) &= e^{-uz}\left[\sum_{s=0}^{m} (-)^s \frac{A_s^*(\theta,z)}{u^s} + O\left(\frac{1}{u^{m+1}}\right)\right] \\ \frac{d}{dz}W_2(u,\theta,z) &= -u\,e^{-uz}\left[\sum_{s=0}^{m} (-)^s \frac{B_s^*(\theta,z)}{u^s} + O\left(\frac{1}{u^{m+1}}\right)\right] \end{aligned}\right\} \quad (z \in \mathbf{H}_2(\theta), \theta \in \mathbf{\Theta}), \tag{2·18}$$

as $u \to +\infty$, *where each of the O's is uniform with respect to z and* θ. *Here m is an arbitrary positive integer or zero, and* $W_1(u, \theta, z)$, $W_2(u, \theta, z)$ *are independent of m.*

The regions of validity $\mathbf{H}_j(\theta)$ $(j = 1, 2)$ are defined as follows. We suppose $a_j(\theta)$ $(j = 1, 2)$ to be any prescribed point of $\mathbf{G}(\theta)$, or the point at infinity on a straight line $\mathscr{L}$ lying in $\mathbf{G}(\theta)$. If $a_j(\theta)$ is at infinity we suppose that $|\arg\{-a_j(\theta)\}| < \frac{1}{2}\pi$ $(j = 1)$ and $|\arg a_j(\theta)| < \frac{1}{2}\pi$ $(j = 2)$. Then $\mathbf{H}_j(\theta)$ comprises those points z of $\mathbf{G}(\theta)$ which can be joined to $a_j(\theta)$ by a path $\mathscr{P}$ having the following properties, t being a typical point of $\mathscr{P}$.

(i) $\mathscr{P}$ lies in $\mathbf{G}(\theta)$.

(ii) $\mathscr{P}$ comprises a finite number of Jordan arcs, each with parametric equations of the form $t = t(\tau)$, where τ is the real parameter of the arc; $t''(\tau)$ is continuous and $t'(\tau)$ does not vanish. If $a_j(\theta)$ is at infinity, $\mathscr{P}$ coincides with $\mathscr{L}$ for all sufficiently large $|\,t\,|$.

(iii) $$\int_{\mathscr{P}} \frac{|\,dt\,|}{1+|\,t\,|^{1+\sigma}} < k. \tag{2·19}$$

(iv) As t traverses $\mathscr{P}$ from $a_j(\theta)$ to z, Re t is monotonic increasing if $j = 1$, and monotonic decreasing if $j = 2$.

3. Remarks on theorem A

First, the common circumstances in which some of the conditions given in §2 are fulfilled are noted.

(i) Condition (2·15) is relatively weak. It is satisfied, for example, if the path joining $c(\theta)$ and z is composed entirely of straight lines, the total number of which is a bounded function of z and θ.

The parametric equation of a typical straight line may be expressed in the form

$$t = (\tau + i\lambda_1)\,e^{i\lambda_2} \quad (\lambda_3 \leqslant \tau \leqslant \lambda_4), \tag{3·1}$$

where λ_1, λ_2, λ_3 and λ_4 are real constants which may depend on θ. Clearly $|t| \geqslant |\tau|$ and

$$\int \frac{|\mathrm{d}t|}{1+|t|^{1+\sigma}} \leqslant \int_{\lambda_3}^{\lambda_4} \frac{\mathrm{d}\tau}{1+|\tau|^{1+\sigma}} \leqslant 2\int_0^\infty \frac{\mathrm{d}\tau}{1+\tau^{1+\sigma}} = k, \tag{3·2}$$

the left-hand integral being taken over the straight line (3·1).

(ii) Similarly, if $\mathscr{P}$ is composed entirely of straight lines, the total number of which is a bounded function of z and θ, then condition (iii) on $\mathscr{P}$ is satisfied. In this event, moreover, condition (ii) on $\mathscr{P}$ is obviously satisfied as well.

We may note in passing that condition (ii) on $\mathscr{P}$ demands rather more than that each arc of $\mathscr{P}$ be a *regular* Jordan arc, for which only $t'(\tau)$ need be continuous.

(iii) It may happen that the inequality (2·2) holds only for $m \leqslant M$, where M is a positive integer or zero and independent of u and θ. In this event it can be deduced from the proof of theorem A given in §§ 5 to 8 that the relations (2·17) and (2·18) hold provided that $m \leqslant M$.

(iv) The method of proof of theorem A, given in §§ 5 to 8, is not immediately applicable to the extended case in which the asymptotic expansion (2·3) contains the additional term $uf_{-1}(\theta, z)$. Langer (1949) has derived formal series solutions of the differential equation in this case and indicated their asymptotic nature when z is real and bounded. An extension of Langer's result to the equation

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = u^{2p} f w; \qquad f \sim 1 + \frac{f_1}{u} + \frac{f_2}{u^2} + \dots,$$

where p is a positive integer, has been given by Erdélyi (1956), again for real bounded z.

4. Lemma on asymptotic sums

We shall require the following result.

Lemma. *Let $\phi_0(\theta)$, $\phi_1(\theta)$ ... be a set of functions satisfying the condition*

$$|\phi_s(\theta)| < c_s \quad (\theta \in \mathbf{\Theta}, s = 0, 1, \dots), \tag{4·1}$$

where $c_0, c_1, \dots$ are independent of u and θ. Then there exists a function $\Phi(u, \theta)$ with the property

$$\Phi(u, \theta) \sim \sum_{s=0}^{\infty} \frac{\phi_s(\theta)}{u^s} \tag{4·2}$$

as $u \to +\infty$, uniformly valid with respect to θ.

This result is known from the general theory of asymptotic sums (van der Corput 1956, p. 391). It suffices here to remark that the function

$$\Phi(u, \theta) = \sum_{s=0}^{\nu(u)} \frac{\phi_s(\theta)}{u^s}, \tag{4·3}$$

where $\nu(u)$ is the largest integer such that

$$c_0 + c_1 + \dots + c_{\nu(u)} + \nu(u) \leqslant u, \tag{4·4}$$

has the required property.

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5. Bounds for the coefficients

Inequality (2·2), with $m = 0$, means that

$$|f(u,\theta,z)| < \frac{k}{1+|z|^{1+\sigma}} \quad (z\in \mathbf{D}(\theta), u \geqslant u_0), \tag{5·1}$$

and for $m \geqslant 1$ the same inequality establishes the equation

$$f_{m-1}(\theta,z) = u_0^{m-1}\left\{f(u_0,\theta,z) - \sum_{s=0}^{m-2}\frac{f_s(\theta,z)}{u_0^s}\right\} + \frac{\varpi k_m}{(1+|z|^{1+\sigma})\,u_0}, \tag{5·2}$$

where ϖ is a number such that $|\varpi| < 1$. If $m = 1$ the sum is absent, and with the aid of (5·1) we deduce that

$$|f_0(\theta,z)| < \frac{k}{1+|z|^{1+\sigma}} \quad (z\in \mathbf{D}(\theta)). \tag{5·3}$$

From these three results we readily show by induction that

$$|f_s(\theta,z)| < \frac{k_s}{1+|z|^{1+\sigma}} \quad (z\in \mathbf{D}(\theta), s = 0, 1, \ldots), \tag{5·4}$$

where k_s is the generic symbol denoting a quantity independent of z, θ and u, but which may depend on s.

We use these results to establish the following properties of the set of coefficients $A_s(\theta, z)$ defined by (2·10).

Lemma

$$|A_s(\theta,z)| < k_s \qquad |A_s'(\theta,z)| < \frac{k_s}{1+|z|^{1+\sigma}} \qquad |A_s''(\theta,z)| < \frac{k_s}{1+|z|^{1+\sigma}} \quad (z\in \mathbf{G}(\theta)). \tag{5·5}$$

This, also, may be proved by induction. Let $d(>0)$ be the shortest distance between the boundaries of $\mathbf{D}(\theta)$ and $\mathbf{G}(\theta)$ for all θ (cf. §2, condition (i) on $\mathbf{G}(\theta)$). We define $\mathbf{G}(\theta,\delta)$ $(0<\delta<d)$ to be $\mathbf{G}(\theta)$ plus the aggregate of all points whose distance from a boundary point of $\mathbf{G}(\theta)$ does not exceed δ. Clearly $\mathbf{G}(\theta,\delta)$ is contained in $\mathbf{D}(\theta)$.

We note that if $z\in \mathbf{G}(\theta,\delta)$ there exists a path joining $c(\theta)$ and z, lying in $\mathbf{G}(\theta,\delta)$ and having the property

$$\int_{c(\theta)}^{z} \frac{|\mathrm{d}t|}{1+|t|^{1+\sigma}} < k. \tag{5·6}$$

For if $z\in \mathbf{G}(\theta)$ this is one of the conditions on $\mathbf{G}(\theta)$. If z is any other point of $\mathbf{G}(\theta,\delta)$ then z_0 exists on the boundary of $\mathbf{G}(\theta)$ such that $|z-z_0|\leqslant\delta$; we take the path in (5·6) to be the known path joining $c(\theta)$ and z_0, plus the join of z_0 and z.

Let us suppose, temporarily, that

$$|A_n'(\theta,z)| < \frac{k_n}{1+|z|^{1+\sigma}} \quad (z\in \mathbf{G}(\theta,\delta), n = 0, 1, \ldots, s). \tag{5·7}$$

Clearly

$$A_n(\theta,z) = \int_{c(\theta)}^{z} A_n'(\theta,t)\,\mathrm{d}t + A_n\{\theta, c(\theta)\}. \tag{5·8}$$

If $z\in \mathbf{G}(\theta,\delta)$ we may take the path of integration to be that of the integral on the left of (5·6). Then substituting (5·7) and the first of (2·16) in (5·8) and using (5·6), we find that

$$|A_n(\theta,z)| < k_n\int_{c(\theta)}^{z} \frac{|\mathrm{d}t|}{1+|t|^{1+\sigma}} + k_n = kk_n + k_n = k_n. \tag{5·9}$$

Let η be an arbitrary number, $0<\eta<\delta$. Then $\mathbf{G}(\theta,\eta)$ is contained in $\mathbf{G}(\theta,\delta)$. Cauchy's integral formula shows that

$$A_s''(\theta,z)=\frac{1}{2\pi i}\int_{|t-z|=\delta-\eta}\frac{A_s'(\theta,t)}{(t-z)^2}\,dt. \tag{5·10}$$

If, now, $z\in\mathbf{G}(\theta,\eta)$, then $t\in\mathbf{G}(\theta,\delta)$. [This is obviously true if $z\in\mathbf{G}(\theta)$. If z is any other point of $\mathbf{G}(\theta,\eta)$ then there is a point z_0 on the boundary of $\mathbf{G}(\theta)$ such that $|z-z_0|<\eta$. Hence $|t-z_0|=|(t-z)+(z-z_0)|<\delta-\eta+\eta=\delta$.] We also have

$$\frac{1+|z|^{1+\sigma}}{1+|t|^{1+\sigma}}\leqslant\frac{1+(|t|+\delta-\eta)^{1+\sigma}}{1+|t|^{1+\sigma}}<k. \tag{5·11}$$

Substituting (5·7) with $n=s$, in (5·10) and using (5·11), we deduce that

$$|A_s''(\theta,z)|<\frac{k_s}{2\pi}\int_{|t-z|=\delta-\eta}\frac{|dt|}{|t-z|^2(1+|t|^{1+\sigma})}$$
$$<\frac{kk_s}{2\pi(1+|z|^{1+\sigma})}\int_{|t-z|=\delta-\eta}\frac{|dt|}{|t-z|^2}=\frac{kk_s}{(\delta-\eta)(1+|z|^{1+\sigma})}, \tag{5·12}$$

provided that $z\in\mathbf{G}(\theta,\eta)$. Finally, substituting this result, (5·9) and (5·4) in (2·9), we obtain

$$|2A_{s+1}'(\theta,z)|<\frac{(\delta-\eta)^{-1}kk_s+k_0k_s+k_1k_{s-1}+\ldots+k_sk_0}{1+|z|^{1+\sigma}}=\frac{k_{s+1}}{1+|z|^{1+\sigma}}\quad(z\in\mathbf{G}(\theta,\eta)).$$

The truth of the lemma is now evident.

6. Equation satisfied by the truncated series

We define

$$L_m(u,\theta,z)=e^{uz}\sum_{s=0}^{m}\frac{A_s(\theta,z)}{u^s}. \tag{6·1}$$

Differentiating with respect to z and suppressing arguments on the right-hand side, we obtain

$$\frac{d}{dz}L_m(u,\theta,z)=u\,e^{uz}\left(\sum_{s=0}^{m}\frac{B_s}{u^s}+\frac{A_m'}{u^{m+1}}\right), \tag{6·2}$$

$$\frac{d^2}{dz^2}L_m(u,\theta,z)=u^2e^{uz}\left(\sum_{s=0}^{m}\frac{C_s}{u^s}+\frac{2A_m'+A_{m-1}''}{u^{m+1}}+\frac{A_m''}{u^{m+2}}\right). \tag{6·3}$$

Hence we have

$$\frac{d^2}{dz^2}L_m(u,\theta,z)-\{u^2+f(u,\theta,z)\}L_m(u,\theta,z)=e^{uz}R_m(u,\theta,z), \tag{6·4}$$

where
$$R_m(u,\theta,z)=u^2\left\{\sum_{s=0}^{m}\frac{C_s}{u^s}+\frac{2A_m'+A_{m-1}''}{u^{m+1}}+\frac{A_m''}{u^{m+2}}\right\}-\{u^2+f(u,\theta,z)\}\sum_{s=0}^{m}\frac{A_s}{u^s}, \tag{6·5}$$

and is a regular function of z in $\mathbf{D}(\theta)$.

The inequality (2·2) implies that

$$f(u,\theta,z)=\sum_{s=0}^{m-1}\frac{f_s}{u^s}+\frac{\varpi k_m}{(1+|z|^{1+\sigma})\,u^m}\quad(z\in\mathbf{D}(\theta)), \tag{6·6}$$

when $u\geqslant u_0$, where $|\varpi|<1$. Substituting (6·6) in (6·5), we find that the coefficients of $u^2, u, \ldots, u^{-m+1}$, vanish as a consequence of (2·8) and (2·9), and with the aid also of (5·4) and (5·5) we easily show that

$$|R_m(u,\theta,z)|<\frac{1}{1+|z|^{1+\sigma}}\frac{k_m}{u^m}\quad(z\in\mathbf{G}(\theta)), \tag{6·7}$$

provided that $u\geqslant u_0$; henceforth we accept this restriction.

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7. Proof of theorem A: $a_j(\theta)$ finite

A proof only for the case $j = 1$ will be recorded. An exactly similar proof holds for $j = 2$; alternatively, the result for $j = 2$ can be deduced from that for $j = 1$ by means of the transformation $z' = -z$.

In this section we suppose that $a_1 \equiv a_1(\theta)$, defined in § 2, is a point of $\mathbf{G}(\theta)$ which is not at infinity. The necessary modifications when a_1 is at infinity are considered in § 8.

From (5·5) and the first of (2·7), we see that

$$|A_s(\theta, a_1)| < k_s, \qquad |B_s(\theta, a_1)| < k_s. \tag{7·1}$$

The lemma of § 4 shows that functions $A(u, \theta)$ and $B(u, \theta)$ exist having the asymptotic expansions

$$A(u,\theta) \sim \sum_{s=0}^{\infty} \frac{A_s(\theta, a_1)}{u^s}, \qquad B(u,\theta) \sim \sum_{s=0}^{\infty} \frac{B_s(\theta, a_1)}{u^s}, \tag{7·2}$$

as $u \to +\infty$, uniformly valid with respect to θ. We define $W_1(u, \theta, z)$ to be the solution of the differential equation (2·1) satisfying the conditions

$$W_1(u,\theta,a_1) = \mathrm{e}^{ua_1} A(u,\theta), \qquad \left[\frac{\mathrm{d}}{\mathrm{d}z} W_1(u,\theta,z)\right]_{z=a_1} = u\,\mathrm{e}^{ua_1} B(u,\theta). \tag{7·3}$$

From (2·1) and (6·4) we obtain

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}\{W_1(z) - L_m(z)\} - \{u^2 + f(z)\}\{W_1(z) - L_m(z)\} = -\mathrm{e}^{uz} R_m(z), \tag{7·4}$$

where $W_1(z) = W_1(u, \theta, z)$ and the arguments u and θ of the other functions have similarly been suppressed.

Let z_1 be a point of the region $\mathbf{H}_1(\theta)$ defined in § 2, and let ζ be a typical point of the path $\mathscr{P}$ joining z_1 and a_1. In order to establish the asymptotic properties (2·17) we solve, by successive approximation, an integral equation equivalent to the differential equation (7·4). We introduce a sequence of functions $h_{m,n}(u, \theta, z_1, \zeta) \equiv h_{m,n}(\zeta)$ defined by $h_{m,0}(\zeta) = 0$ and

$$h_{m,n}(\zeta) = \frac{1}{2u}\int_{a_1}^{\zeta} \{\mathrm{e}^{u(\zeta-t)} - \mathrm{e}^{u(t-\zeta)}\}\{f(t)\,h_{m,n-1}(t) - \mathrm{e}^{ut}R_m(t)\}\,\mathrm{d}t + \alpha_m \mathrm{e}^{u\zeta} + \beta_m \mathrm{e}^{-u\zeta} \quad (n \geqslant 1), \tag{7·5}$$

where the path of integration is the part of $\mathscr{P}$ between ζ and a_1, and α_m, β_m are determined by the conditions

$$h_{m,n}(a_1) = W_1(a_1) - L_m(a_1), \qquad h'_{m,n}(a_1) = W'_1(a_1) - L'_m(a_1). \tag{7·6}$$

By differentiating (7·5), we readily verify that

$$(\mathrm{d}^2/\mathrm{d}\zeta^2)\,h_{m,n}(\zeta) - u^2 h_{m,n}(\zeta) = f(\zeta)\,h_{m,n-1}(\zeta) - \mathrm{e}^{u\zeta}R_m(\zeta) \quad (n \geqslant 1) \tag{7·7}$$

(cf. (7·4)).

The quantities α_m and β_m are independent of n; in fact from (7·5), (7·6), (7·3), (6·1) and (6·2) we obtain

$$\left.\begin{aligned} \alpha_m \mathrm{e}^{ua_1} + \beta_m \mathrm{e}^{-ua_1} &= \mathrm{e}^{ua_1}\left\{A(u,\theta) - \sum_{s=0}^{m} \frac{A_s(\theta,a_1)}{u^s}\right\} = \mathrm{e}^{ua_1} O(u^{-m-1}), \\ \alpha_m \mathrm{e}^{ua_1} - \beta_m \mathrm{e}^{-ua_1} &= \mathrm{e}^{ua_1}\left\{B(u,\theta) - \sum_{s=0}^{m} \frac{B_s(\theta,a_1)}{u^s} - \frac{A'_m(\theta,a_1)}{u^{m+1}}\right\} = \mathrm{e}^{ua_1} O(u^{-m-1}), \end{aligned}\right\} \tag{7·8}$$

where the O's are uniform with respect to θ. Solving these equations, we see that

$$\alpha_m = O(u^{-m-1}), \qquad \beta_m = e^{2ua_1} O(u^{-m-1}). \tag{7·9}$$

Hence

$$|\alpha_m e^{u\zeta}| < k_m u^{-m-1} |e^{u\zeta}|, \qquad |\beta_m e^{-u\zeta}| = |e^{2ua_1 - u\zeta}| O(u^{-m-1}) < k_m u^{-m-1} |e^{u\zeta}|, \tag{7·10}$$

since

$$\operatorname{Re} z_1 \geqslant \operatorname{Re} \zeta \geqslant \operatorname{Re} t \geqslant \operatorname{Re} a_1, \tag{7·11}$$

in consequence of condition (iv) on $\mathscr{P}$, given in § 2.

Next, we have from (7·5)

$$h_{m,1}(\zeta) = -\frac{1}{2u} \int_{a_1}^{\zeta} \{e^{u\zeta} - e^{u(2t-\zeta)}\} R_m(t)\, dt + \alpha_m e^{u\zeta} + \beta_m e^{-u\zeta}. \tag{7·12}$$

From (7·11) we deduce that

$$|e^{u(2t-\zeta)}| \leqslant |e^{u\zeta}|, \tag{7·13}$$

and using this inequality and (6·7), we see that

$$\left| \int_{a_1}^{\zeta} \{e^{u\zeta} - e^{u(2t-\zeta)}\} R_m(t)\, dt \right| < \frac{2k_m}{u^m} |e^{u\zeta}| \int_{\mathscr{P}} \frac{|dt|}{1 + |t|^{1+\sigma}} < \frac{2kk_m}{u^m} |e^{u\zeta}|, \tag{7·14}$$

in virtue of (2·19). Substituting this result and (7·10) in (7·12), we obtain

$$|h_{m,1}(\zeta)| < k_m u^{-m-1} |e^{u\zeta}|. \tag{7·15}$$

From (7·5) we have also

$$h_{m,n+1}(\zeta) - h_{m,n}(\zeta) = \frac{1}{2u} \int_{a_1}^{\zeta} \{e^{u(\zeta-t)} - e^{u(t-\zeta)}\} f(t) \{h_{m,n}(t) - h_{m,n-1}(t)\}\, dt \quad (n \geqslant 1). \tag{7·16}$$

Taking $n = 1$ and substituting (5·1) with z replaced by t, (7·15) with ζ replaced by t and (7·13), we deduce that

$$|h_{m,2}(\zeta) - h_{m,1}(\zeta)| < kk_m u^{-m-2} |e^{u\zeta}|.$$

Continuing the argument by induction, we establish the key result

$$|h_{m,n+1}(\zeta) - h_{m,n}(\zeta)| < k_m k^n u^{-m-n-1} |e^{u\zeta}| \quad (n = 0, 1, \ldots), \tag{7·17}$$

where k_m and k are of course independent of n as well as of u, θ, z and ζ.

Thus the series

$$T = \sum_{n=0}^{\infty} \{h_{m,n+1}(\zeta) - h_{m,n}(\zeta)\}$$

converges if, as we shall now suppose, $u > k$. We shall prove that its sum is $W_1(\zeta) - L_m(\zeta)$. In the method of proof used previously (Olver 1954*a*, § 10; 1956, §§ 12 and 13) this was achieved quite easily by tacitly supposing that the region corresponding to $\mathbf{H}_1(\theta)$ is, in fact, a complex domain, i.e. an open connex set of points. Term-by-term differentiation of the series T is then valid within the domain, and it follows immediately from (7·7) that T satisfies the same differential equation (7·4) as $W_1(z) - L_m(z)$. Therefore, because of the boundary conditions (7·6) imposed at a_1, the two functions are equal in $\mathbf{H}_1(\theta)$. The present proof does not depend on assumptions of this kind concerning the nature of the region $\mathbf{H}_1(\theta)$.

Differentiating (7·12) and (7·16) with respect to ζ, we find that

$$h'_{m,1}(\zeta) = -\frac{1}{2} \int_{a_1}^{\zeta} \{e^{u\zeta} + e^{u(2t-\zeta)}\} R_m(t)\, dt + \alpha_m u\, e^{u\zeta} - \beta_m u\, e^{-u\zeta}, \tag{7·18}$$

and

$$h'_{m,n+1}(\zeta) - h'_{m,n}(\zeta) = \frac{1}{2} \int_{a_1}^{\zeta} \{e^{u(\zeta-t)} + e^{u(t-\zeta)}\} f(t) \{h_{m,n}(t) - h_{m,n-1}(t)\}\, dt \quad (n \geqslant 1). \tag{7·19}$$

Using (2·19), (5·1), (6·7), (7·10), (7·13) and (7·17), we may show that

$$|h'_{m,n+1}(\zeta)-h'_{m,n}(\zeta)|<k_m k^n u^{-m-n}|e^{u\zeta}| \quad (n=0,1,\ldots). \tag{7·20}$$

And from (7·17), (6·7), (5·1) and the relations

$$h''_{m,1}(\zeta)=u^2h_{m,1}(\zeta)-e^{u\zeta}R_m(\zeta), \tag{7·21}$$

$$h''_{m,n+1}(\zeta)-h''_{m,n}(\zeta)=u^2\{h_{m,n+1}(\zeta)-h_{m,n}(\zeta)\}+f(\zeta)\{h_{m,n}(\zeta)-h_{m,n-1}(\zeta)\} \quad (n\geqslant 1), \tag{7·22}$$

obtained from (7·7), we deduce that

$$|h''_{m,n+1}(\zeta)-h''_{m,n}(\zeta)|<k_m k^n u^{-m-n+1}|e^{u\zeta}| \quad (n=0,1,\ldots). \tag{7·23}$$

Thus the series $\sum_{n=0}^{\infty}\{h''_{m,n+1}(\zeta)-h''_{m,n}(\zeta)\}$ converges, and from (7·21), (7·22) it is seen that

$$\sum_{n=0}^{\infty}\{h''_{m,n+1}(\zeta)-h''_{m,n}(\zeta)\}=\{u^2+f(\zeta)\}\sum_{n=0}^{\infty}\{h_{m,n+1}(\zeta)-h_{m,n}(\zeta)\}-e^{u\zeta}R_m(\zeta). \tag{7·24}$$

Condition (ii) on $\mathscr{P}$, given in § 2, affirms that $\mathscr{P}$ has a parametric equation of the form

$$\zeta=\zeta(\tau) \quad (\tau_0\leqslant\tau\leqslant\tau_1), \tag{7·25}$$

where τ_0 and τ_1 correspond to a_1 and z_1, respectively. The function $\zeta(\tau)$ is continuous and $\zeta'(\tau)$, $\zeta''(\tau)$ have only a finite number of discontinuities in the interval $\tau_0\leqslant\tau\leqslant\tau_1$. The series

$$S(\tau)=\sum_{n=0}^{\infty}[h''_{m,n+1}\{\zeta(\tau)\}-h''_{m,n}\{\zeta(\tau)\}] \tag{7·26}$$

is accordingly a uniformly convergent series of continuous functions when $\tau_0\leqslant\tau\leqslant\tau_1$, and we may multiply both sides of this equation by $\zeta'(\tau)$ and integrate term by term to obtain

$$\int_{\tau_0}^{\tau}S(\tau)\,\zeta'(\tau)\,d\tau+h'_{m,1}(a_1)=\sum_{n=0}^{\infty}[h'_{m,n+1}\{\zeta(\tau)\}-h'_{m,n}\{\zeta(\tau)\}]. \tag{7·27}$$

Similarly

$$\int_{\tau_0}^{\tau}\left\{\int_{\tau_0}^{\tau}S(\tau)\,\zeta'(\tau)\,d\tau+h'_{m,1}(a_1)\right\}\zeta'(\tau)\,d\tau+h_{m,1}(a_1)=\sum_{n=0}^{\infty}[h_{m,n+1}\{\zeta(\tau)\}-h_{m,n}\{\zeta(\tau)\}]=T(\tau), \tag{7·28}$$

say. Differentiation of the last equation with respect to τ yields

$$\begin{aligned}T''(\tau)-\{\zeta''(\tau)/\zeta'(\tau)\}\,T'(\tau)&=\{\zeta'(\tau)\}^2S(\tau)\\&=\{\zeta'(\tau)\}^2[u^2+f\{\zeta(\tau)\}]\,T(\tau)-\{\zeta'(\tau)\}^2e^{u\zeta(\tau)}R_m\{\zeta(\tau)\},\end{aligned} \tag{7·29}$$

on substituting for $S(\tau)$ by means of (7·26) and (7·24).

Equation (7·29) is a second-order differential equation for $T(\tau)$. Since, by hypothesis, $\zeta'(\tau)$ does not vanish, all the coefficients are continuous except at the discontinuities of $\zeta'(\tau)$ and $\zeta''(\tau)$, i.e. the junctions of the Jordan arcs. Substituting $z=\zeta(\tau)$ in (7·4), we see that the function $W_1\{\zeta(\tau)\}-L_m\{\zeta(\tau)\}$ satisfies exactly the same differential equation. Moreover, from (7·6) we note that at $\tau=\tau_0$, i.e. at $\zeta=a_1$, $T(\tau)$ and $T'(\tau)$ are respectively equal to the values of $W_1\{\zeta(\tau)\}-L_m\{\zeta(\tau)\}$ and its τ-derivative. Since a linear differential equation with continuous coefficients has a unique solution corresponding to given initial conditions (Ince 1944, § 3·32), it follows that

$$T(\tau)=W_1\{\zeta(\tau)\}-L_m\{\zeta(\tau)\}, \qquad T'(\tau)=(d/d\tau)\,[W_1\{\zeta(\tau)\}-L_m\{\zeta(\tau)\}], \tag{7·30}$$

along the arc with terminal point τ_0. At the junction of this arc with the next, $T(\tau)$ is continuous but $T'(\tau)$ is discontinuous. However, $T'(\tau)/\zeta'(\tau)$ and $W_1'\{\zeta(\tau)\}-L_m'\{\zeta(\tau)\}$ are both continuous. Accordingly, equations (7·30) hold across the junction and therefore also along the second arc. Extending the argument, we see that equations (7·30) hold along the whole of $\mathscr{P}$.

In particular, at $\tau=\tau_1$, i.e. at $\zeta=z_1$, we have

$$\left.\begin{aligned} W_1(z_1)-L_m(z_1) &= T(\tau_1) = \sum_{n=0}^{\infty}\{h_{m,n+1}(z_1)-h_{m,n}(z_1)\},\\ W_1'(z_1)-L_m'(z_1) &= T'(\tau_1)/\zeta'(\tau_1) = \sum_{n=0}^{\infty}\{h'_{m,n+1}(z_1)-h'_{m,n}(z_1)\}.\end{aligned}\right\} \tag{7·31}$$

If we suppose that $u>2k$, we deduce immediately from these results and (7·17), (7·20) that

$$\left.\begin{aligned} |W_1(z_1)-L_m(z_1)| &< k_m u^{-m-1}|e^{uz_1}|\sum_{n=0}^{\infty}2^{-n} = 2k_m u^{-m-1}|e^{uz_1}|,\\ |W_1'(z_1)-L_m'(z_1)| &< k_m u^{-m}|e^{uz_1}|\sum_{n=0}^{\infty}2^{-n} = 2k_m u^{-m}|e^{uz_1}|.\end{aligned}\right\} \tag{7·32}$$

These are the desired results.

8. Proof of theorem A: $a_j(\theta)$ at infinity

We now suppose that $a_1\equiv a_1(\theta)$ is the point at infinity on a straight line $\mathscr{L}$ lying in $\mathbf{G}(\theta)$. Relations (5·8) and (5·5) show that $A_s(\theta,z)$ tends to a limit $A_s(\theta,a_1)$, say, as $z\to a_1$ along $\mathscr{L}$, and that

$$|A_s(\theta,a_1)|\leqslant k_s. \tag{8·1}$$

Therefore, by the lemma of § 4, a function $A(u,\theta)$ exists having the asymptotic expansion

$$A(u,\theta)\sim\sum_{s=0}^{\infty}\frac{A_s(\theta,a_1)}{u^s}, \tag{8·2}$$

as $u\to+\infty$, uniformly valid with respect to θ.

We define $W_1(u,\theta,z)\equiv W_1(z)$ to be the solution of the differential equation (2·1) with the properties

$$\lim\{e^{-uz}W_1(u,\theta,z)\}=A(u,\theta),\qquad \lim\{e^{-uz}(d/dz)\,W_1(u,\theta,z)\}=uA(u,\theta), \tag{8·3}$$

as $z\to a_1$ along $\mathscr{L}$. At this stage we do not know whether such a solution necessarily exists. There can be at most one such solution, however. For suppose $W_1^*(z)$ were a second solution with the same properties. Since $|\arg(-a_1)|<\frac{1}{2}\pi$ (cf. § 2), it follows that W_1, W_1^*, W_1' and $W_1^{*\prime}$ all tend to zero as $z\to a_1$; accordingly the Wronskian of W_1 and W_1^* vanishes.

We define the sequence $h_{m,n}(\zeta)$ by $h_{m,0}(\zeta)=0$ and

$$h_{m,n}(\zeta)=\frac{1}{2u}\int_{a_1}^{\zeta}\{e^{u(\zeta-t)}-e^{u(t-\zeta)}\}\{f(t)\,h_{m,n-1}(t)-e^{ut}R_m(t)\}\,dt+\alpha_m e^{u\zeta}\quad(n\geqslant 1) \tag{8·4}$$

(cf. (7·5)), where

$$\alpha_m=A(u,\theta)-\sum_{s=0}^{m}\frac{A_s(\theta,a_1)}{u^s}. \tag{8·5}$$

Here ζ is again a typical point of the path $\mathscr{P}$ joining a_1 and z_1, and the path of integration in (8·4) is the part of $\mathscr{P}$ between ζ and a_1; it coincides with $\mathscr{L}$ for all sufficiently large $|\zeta|$ (cf. § 2, condition (ii)).

Equation (8·4) corresponds to (7·5) and the analysis of § 7 between (7·5) and (7·24) can be repeated here with trivial changes; in particular the relations (7·17) and (7·20) to (7·24) are again satisfied.

The parametric equation of $\mathscr{P}$ is of the form (7·25) but with $\tau_0 = -\infty$ of necessity, since $\zeta(\tau)$ is continuous and therefore finite with τ. Term-by-term integration of a uniformly convergent series over an infinite interval is permissible if the same operation on the moduli of the terms yields a convergent series (Titchmarsh 1939, § 1·77). If $u>k$ this condition is satisfied by the series on the right of (7·26) multiplied by $\zeta'(\tau)$, because $\int_{-\infty}^{\tau} | e^{u\zeta(\tau)}\zeta'(\tau) | d\tau$ converges. Therefore we have

$$\int_{-\infty}^{\tau} S(\tau)\,\zeta'(\tau)\,d\tau = \sum_{n=0}^{\infty}[h'_{m,\,n+1}\{\zeta(\tau)\}-h'_{m,\,n}\{\zeta(\tau)\}], \tag{8·6}$$

corresponding to (7·27), and similarly

$$\int_{-\infty}^{\tau}\left\{\int_{-\infty}^{\tau} S(\tau)\,\zeta'(\tau)\,d\tau\right\}\zeta'(\tau)\,d\tau = \sum_{n=0}^{\infty}[h_{m,\,n+1}\{\zeta(\tau)\}-h_{m,\,n}\{\zeta(\tau)\}] = T(\tau), \tag{8·7}$$

corresponding to (7·28). Differentiation of (8·7) again yields (7·29).

We now examine the behaviour of $T(\tau)$ as $\tau\to-\infty$. Since $\mathscr{P}$ coincides with the straight line $\mathscr{L}$ for all sufficiently large $|\zeta|$, it follows that if ϵ is an arbitrary positive number then

$$\int_{a_1}^{\zeta(\tau)}\frac{|dt|}{1+|t|^{1+\sigma}}<\epsilon, \tag{8·8}$$

for all $\tau\leqslant\tau(\epsilon)$ (assignable). With this restriction, we derive from (8·4), with $n=1$,

$$|h_{m,1}(\zeta)-\alpha_m e^{u\zeta}| = \left|\frac{1}{2u}\int_{a_1}^{\zeta}\{e^{u\zeta}-e^{u(2t-\zeta)}\}R_m(t)\,dt\right| \leqslant \frac{\epsilon k_m}{u^{m+1}}|e^{u\zeta}|, \tag{8·9}$$

in consequence of (6·7), (7·13) and (8·8). Similarly, if $n\geqslant 1$ we have

$$\begin{aligned}|h_{m,\,n+1}(\zeta)-h_{m,\,n}(\zeta)| &= \left|\frac{1}{2u}\int_{a_1}^{\zeta}\{e^{u(\zeta-t)}-e^{u(t-\zeta)}\}f(t)\,\{h_{m,\,n}(t)-h_{m,\,n-1}(t)\}\,dt\right|\\ &< \epsilon k_m k^n u^{-m-n-1}|e^{u\zeta}|,\end{aligned} \tag{8·10}$$

in consequence of (5·1), (7·13), (7·17) and (8·8). Therefore, if $u>2k$, it follows that

$$\left|\sum_{n=0}^{\infty}\{h_{m,\,n+1}(\zeta)-h_{m,\,n}(\zeta)\}-\alpha_m e^{u\zeta}\right| < 2\epsilon k_m u^{-m-1}|e^{u\zeta}|. \tag{8·11}$$

Again, equations (7·18) and (7·19) still hold here provided that we set $\beta_m = 0$ in the former, and from them we deduce that

$$\left|\sum_{n=0}^{\infty}\{h'_{m,\,n+1}(\zeta)-h'_{m,\,n}(\zeta)\}-\alpha_m u\,e^{u\zeta}\right| < 2\epsilon k_m u^{-m}|e^{u\zeta}| \tag{8·12}$$

(cf. (8·11)). The inequalities (8·11) and (8·12) show that

$$e^{-u\zeta(\tau)}T(\tau)\to\alpha_m, \qquad e^{-u\zeta(\tau)}\{T'(\tau)/\zeta'(\tau)\}\to\alpha_m u, \tag{8·13}$$

as $\tau\to-\infty$.

Let $W_1^*(z)$ be the solution of the differential equation (2·1) satisfying the conditions

$$W_1^*(z_1) = T(\tau_1)+L_m(z_1), \qquad W_1^{*\prime}(z_1) = \{T'(\tau_1)/\zeta'(\tau_1)\}+L_m'(z_1), \tag{8·14}$$

Then $W_1^*\{\zeta(\tau)\}-L_m\{\zeta(\tau)\}$ satisfies the differential equation (7·29), and since at $\tau = \tau_1$ this function and its τ-derivative are respectively equal to $T(\tau)$ and $T'(\tau)$ it follows as in § 7 that

$$T(\tau) = W_1^*\{\zeta(\tau)\}-L_m\{\zeta(\tau)\}, \qquad T'(\tau) = (\mathrm{d}/\mathrm{d}\tau)\,[W_1^*\{\zeta(\tau)\}-L_m\{\zeta(\tau)\}], \tag{8·15}$$

for all τ in the range $-\infty<\tau\leqslant\tau_1$ (cf. (7·30)).

Letting $\tau\to-\infty$, we obtain from these equations and (8·13)

$$\left.\begin{aligned}\lim\{\mathrm{e}^{-u\zeta}W_1^*(\zeta)\} &= \alpha_m+\lim\{\mathrm{e}^{-u\zeta}L_m(\zeta)\} = A(u,\theta),\\ \lim\{\mathrm{e}^{-u\zeta}W_1^{*\prime}(\zeta)\} &= \alpha_m u+\lim\{\mathrm{e}^{-u\zeta}L_m'(\zeta)\} = uA(u,\theta),\end{aligned}\right\} \tag{8·16}$$

on substituting (8·5), (6·1), (6·2), and using the equations $A_s'(\theta,a_1) = 0$, $B_s(\theta,a_1) = A_s(\theta,a_1)$, obtained from (5·5) and the first of (2·7).

Referring to (8·3), we now see that the solution $W_1(z)$ does exist; in fact, $W_1(z) = W_1^*(z)$. Equations (8·15) show that equations (7·31) again hold, and from them we deduce the inequalities (7·32) as in § 7. This completes the proof.

PART 2. CASE B

9. Statement of conditions and theorem B

Where possible we shall shorten statements and proofs by reference to the corresponding results for case A given in part 1.

The standard form of differential equation for this case is

$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2} = \{u^2z+f(u,\theta,z)\}\,w \tag{9·1}$$

(cf. (1·9)), where (as in part 1) u is a large positive parameter, $\theta\in\mathbf{\Theta}$, $z\in\mathbf{D}(\theta)$, a simply-connected complex domain bounded or otherwise. We suppose in addition that $\mathbf{D}(\theta)$ contains the circle $|z|\leqslant b$, where b is positive and independent of u and θ.

We assume that $f(u,\theta,z)$ is a regular function of z in $\mathbf{D}(\theta)$ and that

$$\left|f(u,\theta,z)-\sum_{s=0}^{m-1}\frac{f_s(\theta,z)}{u^s}\right|<\frac{1}{1+|z|^{\frac{1}{2}+\sigma}}\frac{k_m}{u^m} \quad (m = 0,1,2,\ldots) \tag{9·2}$$

(cf. (2·2)), for all $z\in\mathbf{D}(\theta)$, $\theta\in\mathbf{\Theta}$, $u\geqslant u_0$. Each coefficient $f_s(\theta,z)$ is independent of u† and regular in $\mathbf{D}(\theta)$, and the symbols σ and k_m have the meanings assigned in § 2.

Formal solutions of (9·1) in inverse powers of u are given by

$$w = \mathrm{Ci}(u^{\frac{2}{3}}z)\sum_{s=0}^{\infty}\frac{A_s(\theta,z)}{u^s}+\frac{\mathrm{Ci}'(u^{\frac{2}{3}}z)}{u^{\frac{4}{3}}}\sum_{s=0}^{\infty}\frac{B_s(\theta,z)}{u^s} \tag{9·3}$$

(cf. (2·4)), where $\mathrm{Ci}(u^{\frac{2}{3}}z)$ is any solution of the Airy equation

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}\mathrm{Ci}(u^{\frac{2}{3}}z) = u^2z\,\mathrm{Ci}(u^{\frac{2}{3}}z), \tag{9·4}$$

† Except in as much as θ may depend on u.

and the coefficients $A_s(\theta, z)$ and $B_s(\theta, z)$ are to be determined. Term-by-term differentiation of (9·3) and use of (9·4) yields

$$\frac{dw}{dz} = \mathrm{Ci}(u^{\frac{2}{3}}z)\sum_{s=0}^{\infty}\frac{C_s(\theta,z)}{u^s} + u^{\frac{2}{3}}\mathrm{Ci}'(u^{\frac{2}{3}}z)\sum_{s=0}^{\infty}\frac{D_s(\theta,z)}{u^s}, \tag{9·5}$$

$$\frac{d^2w}{dz^2} = u^2\mathrm{Ci}(u^{\frac{2}{3}}z)\sum_{s=0}^{\infty}\frac{E_s(\theta,z)}{u^s} + u^{\frac{4}{3}}\mathrm{Ci}'(u^{\frac{2}{3}}z)\sum_{s=0}^{\infty}\frac{F_s(\theta,z)}{u^s}, \tag{9·6}$$

where
$$C_s = A_s' + zB_s, \qquad D_s = A_s + B_{s-2}', \tag{9·7}$$

$$\left.\begin{aligned} E_s &= C_{s-2}' + zD_s = A_{s-2}'' + zA_s + 2zB_{s-2}' + B_{s-2}, \\ F_s &= C_s + D_s' = 2A_s' + B_{s-2}'' + zB_s. \end{aligned}\right\} \tag{9·8}$$

Equation (9·1) is formally satisfied if

$$\left.\begin{aligned} E_{s+2} &= zA_{s+2} + f_0A_s + f_1A_{s-1} + \dots + f_sA_0, \\ F_{s+2} &= zB_{s+2} + f_0B_s + f_1B_{s-1} + \dots + f_sB_0. \end{aligned}\right\} \tag{9·9}$$

Substituting (9·8) in (9·9), we obtain

$$2zB_s' + B_s = -A_s'' + f_0A_s + f_1A_{s-1} + \dots + f_sA_0, \tag{9·10}$$

$$2A_{s+2}' = -B_s'' + f_0B_s + f_1B_{s-1} + \dots + f_sB_0. \tag{9·11}$$

Integrating these equations, we find that A_0 = constant, A_1 = constant, and

$$B_s = \tfrac{1}{2}z^{-\frac{1}{2}}\int_0^z z^{-\frac{1}{2}}(-A_s'' + f_0A_s + f_1A_{s-1} + \dots + f_sA_0)\,dz, \tag{9·12}$$

$$A_{s+2} = -\tfrac{1}{2}B_s' + \tfrac{1}{2}\int (f_0B_s + f_1B_{s-1} + \dots + f_sB_0)\,dz. \tag{9·13}$$

Except for the presence of arbitrary constants of integration in (9·13) these two equations determine a set of functions $A_s(\theta, z)$ and $B_s(\theta, z)$ which are regular and single-valued in $\mathbf{D}(\theta)$. We note that if the coefficients $f_s(\theta, z)$ of odd suffix all vanish, then we can arrange that the same is true of $A_s(\theta, z)$ and $B_s(\theta, z)$, i.e. the series occurring in (9·3) contain only even powers of u^{-1}; in particular, this happens when $f(u, \theta, z)$ is independent of u.

We prescribe $\mathbf{G}(\theta)$ here to be a closed subdomain of $\mathbf{D}(\theta)$, having the properties;

(i) $\mathbf{G}(\theta)$ contains the circle $|z| \leqslant b$.

(ii) The distance between each boundary point z_0 of $\mathbf{G}(\theta)$ and each boundary point of $\mathbf{D}(\theta)$ is not less than $d/|z_0|^{\frac{1}{2}}$, where d is a positive constant, assignable independently of θ.

(iii) $$\int_0^z \frac{|dt|}{1+|t|^{1+\sigma_1}} < k \quad (z \in \mathbf{G}(\theta)), \tag{9·14}$$

for some path lying wholly in $\mathbf{G}(\theta)$, where k is the generic symbol denoting a number independent of u, θ and z (cf. § 2), and

$$\sigma_1 = \min(\sigma, \tfrac{3}{2}). \tag{9·15}$$

We suppose that the arbitrary constants associated with the determination of the coefficients are such that

$$|A_s\{\theta, c(\theta)\}| < k_s, \tag{9·16}$$

where $c(\theta)$ is a prescribed point of $\mathbf{G}(\theta)$ (cf. (2·16)).

Using the notation

$$P_1(z) = \mathrm{Ai}\,(z), \qquad P_2(z) = \mathrm{Ai}\,(e^{\frac{2}{3}\pi i}z), \qquad P_3(z) = \mathrm{Ai}\,(e^{-\frac{2}{3}\pi i}z), \tag{9·17}$$

for the Airy functions (Olver 1954*a*, equations (4·6)), we can now state the existence theorem:

THEOREM B. *The differential equation* (9·1) *possesses solutions* $W_j(u,\theta,z)$ $(j=1,2,3)$ *with the properties*

$$W_j(u,\theta,z) = P_j(u^{\frac{2}{3}}z)\left[\sum_{s=0}^{m}\frac{A_s(\theta,z)}{u^s}+O\left(\frac{1}{u^{m+1}}\right)\right]+\frac{P_j'(u^{\frac{2}{3}}z)}{u^{\frac{4}{3}}}\left[\sum_{s=0}^{m-1}\frac{B_s(\theta,z)}{u^s}+\frac{1}{1+|z|^{\frac{1}{2}}}O\left(\frac{1}{u^m}\right)\right], \tag{9·18}$$

$$\frac{\mathrm{d}}{\mathrm{d}z}W_j(u,\theta,z) = P_j(u^{\frac{2}{3}}z)\left[\sum_{s=0}^{m-1}\frac{C_s(\theta,z)}{u^s}+(1+|z|^{\frac{1}{2}})\,O\left(\frac{1}{u^m}\right)\right]+u^{\frac{2}{3}}P_j'(u^{\frac{2}{3}}z)\left[\sum_{s=0}^{m}\frac{D_s(\theta,z)}{u^s}+O\left(\frac{1}{u^{m+1}}\right)\right], \tag{9·19}$$

as $u\to+\infty$, *valid when* $z\in\mathbf{H}_j(\theta)$ *and* $\theta\in\mathbf{\Theta}$, *each of the O's being uniform with respect to z and* θ. *Here m is an arbitrary positive integer or zero and* $W_j(u,\theta,z)$ *is independent of m.*

The regions of validity $\mathbf{H}_j(\theta)$ $(j=1,2,3)$ are defined as follows. We take $a_j(\theta)$ to be any prescribed point of $\mathbf{G}(\theta)$, or the point at infinity on a straight line $\mathscr{L}$ lying in $\mathbf{G}(\theta)$. If $a_j(\theta)$ is at infinity we make the restriction

$$|\arg\{\rho_j a_j(\theta)\}|<\tfrac{1}{3}\pi, \qquad \text{where} \qquad \rho_1\equiv 1, \qquad \rho_2\equiv e^{\frac{2}{3}\pi i}, \qquad \rho_3\equiv e^{-\frac{2}{3}\pi i}. \tag{9·20}$$

Then $\mathbf{H}_j(\theta)$ comprises those points z of $\mathbf{G}(\theta)$ which can be joined to $a_j(\theta)$ by a path $\mathscr{P}$ having the following properties; t being a typical point of $\mathscr{P}$.

(i) $\mathscr{P}$ lies in $\mathbf{G}(\theta)$.

(ii) $\mathscr{P}$ comprises a finite number of Jordan arcs, each with a parametric equation of the form $t=t(\tau)$, where τ is the real parameter of the arc; $t''(\tau)$ is continuous and $t'(\tau)$ does not vanish. If $a_j(\theta)$ is at infinity, $\mathscr{P}$ coincides with $\mathscr{L}$ for all sufficiently large $|t|$.

(iii) $$\int_{\mathscr{P}}\frac{|\mathrm{d}t|}{1+|t|^{1+\sigma_1}}<k. \tag{9·21}$$

(iv) As t traverses $\mathscr{P}$ from $a_j(\theta)$ to z, $|\exp\{\frac{2}{3}(\rho_j t)^{\frac{3}{2}}\}|$ is monotonic decreasing, where ρ_j is defined by (9·20).

10. REMARKS ON THEOREM B

In applications of the theorem many of the conditions are conveniently interpreted by transformation to the x-plane, where

$$x=\tfrac{2}{3}z^{\frac{3}{2}}, \tag{10·1}$$

the principal value being taken; examples of this interpretation are given below. It is less convenient, however, to develop the *theory* with x as primary variable in place of z, because the transformed differential equation has a singularity at $x=0$; moreover, it would be necessary to consider the two-sheeted region $|\arg x|\leqslant\frac{3}{2}\pi$.

(i) If the distance between the boundary points of $\mathbf{G}(\theta)$ and $\mathbf{D}(\theta)$ has a positive lower bound d', condition (ii) on $\mathbf{G}(\theta)$ is fulfilled with $d=b^{\frac{1}{2}}d'$; this is a consequence of condition (i) on $\mathbf{G}(\theta)$. Condition (ii) is less restrictive than the corresponding condition given in § 5 of Olver 1954*a*. Effectively the new condition means that the distance between the boundaries of the x-maps of $\mathbf{G}(\theta)$ and $\mathbf{D}(\theta)$ has a positive lower bound, but because of the two-sheeted nature of the x-maps the precise formulation of the condition in this form is complicated.

A typical example of the greater usefulness of the new form of condition occurs in Olver 1954*b* in an application to Bessel functions of large order. If $\mathbf{D}(\theta)$ is taken to be the ζ-domain corresponding to $|\arg z| < \pi$ (1954*b*, equation (4·6) and figures 3 and 6), then the ζ-domain corresponding to $|\arg z| \leqslant \pi - \epsilon < \pi$ fulfils condition (ii) on $\mathbf{G}(\theta)$ but not the conditions of Olver 1954*a*, § 5.

(ii) Condition (iii) on $\mathbf{G}(\theta)$ is satisfied if the path joining 0 and z is composed of straight lines, the total number of which is a bounded function of z and θ. This is clear from § 3 (i).

The same condition is also satisfied if the x-map of the path is composed of a bounded number of straight lines. For the parametric equation of the curve in the z-plane corresponding to a typical straight line in the x-plane may be expressed in the form

$$z = z(\tau) = \{\tfrac{3}{2}(\tau + \mathrm{i}\lambda_1)\, \mathrm{e}^{\mathrm{i}\lambda_2}\}^{\frac{2}{3}} \quad (\lambda_3 \leqslant \tau \leqslant \lambda_4), \tag{10·2}$$

where λ_1, λ_2, λ_3 and λ_4 are real constants. Clearly

$$|z| \geqslant (\tfrac{3}{2}|\tau|)^{\frac{2}{3}} \geqslant |\tau|^{\frac{2}{3}} \quad \text{and} \quad |\mathrm{d}z| = |z|^{-\frac{1}{2}}\,\mathrm{d}\tau \leqslant |\tau|^{-\frac{1}{3}}\,\mathrm{d}\tau.$$

Therefore we have

$$\int \frac{|\mathrm{d}z|}{1+|z|^{1+\sigma_1}} \leqslant \int_{\lambda_3}^{\lambda_4} \frac{\mathrm{d}\tau}{|\tau|^{\frac{1}{3}}(1+|\tau|^{\frac{2}{3}+\frac{2}{3}\sigma_1})} \leqslant 2\int_0^\infty \frac{\mathrm{d}\tau}{\tau^{\frac{1}{3}}(1+\tau^{\frac{2}{3}+\frac{2}{3}\sigma_1})} = k, \tag{10·3}$$

which proves the statement.

(iii) Similarly, condition (iii) on $\mathscr{P}$ is satisfied if $\mathscr{P}$ consists of a bounded number of straight lines, or if the x-map of $\mathscr{P}$ consists of a bounded number of straight lines plus the x-map of $\mathscr{L}$ (should $a_j(\theta)$ be at infinity).

In these circumstances moreover, condition (ii) on $\mathscr{P}$ is satisfied as well. This is immediately clear from (10·2) unless $\lambda_1 = 0$ when $z'(\tau)$ and $z''(\tau)$ each become infinite at $\tau = 0$. In this event, however, the path passes through the origin and the corresponding z-curve comprises two rays emanating from the origin, in directions differing by an angle $\frac{2}{3}\pi$. We have already seen that the condition is satisfied on each of these rays.

(iv) With $j = 1$, condition (iv) on $\mathscr{P}$ implies that in the x-plane $\operatorname{Re} x$ is monotonic decreasing as we proceed along $\mathscr{P}$ away from the map of $a_1(\theta)$. Similar interpretations hold for $j = 2$ and 3.

(v) If the inequality (9·2) holds only for $m \leqslant M$, where M is a positive integer or zero and independent of u and θ, then (9·18) and (9·19) hold unchanged for $m \leqslant M-1$. They also hold for $m = M$, provided that the error terms $O(u^{-m-1})$ and $O(u^{-m})$ in (9·18) are changed to $O(u^{-M-\frac{2}{3}})$ and $O(u^{-M+\frac{2}{3}})$, respectively, and the error terms $O(u^{-m})$ and $O(u^{-m-1})$ in (9·19) are changed to $O(u^{-M+\frac{1}{3}})$ and $O(u^{-M-\frac{2}{3}})$ respectively. This result follows from the inequalities (13·26) given later. Other forms of the error terms are indicated by relations (13·23) and (13·24).

(vi) Langer (1949) has developed formal series solutions of (9·1) in the extended case in which the asymptotic representation (9·2) contains the additional term $uf_{-1}(\theta, z)$, and has established their asymptotic nature for real bounded z.

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11. Bounds for the coefficients

From (9·2) we deduce that

$$|f(u,\theta,z)|<\frac{k}{1+|z|^{\frac{1}{2}+\sigma}}\quad(z\in\mathbf{D}(\theta),u\geqslant u_0),\tag{11·1}$$

and

$$|f_s(\theta,z)|<\frac{k_s}{1+|z|^{\frac{1}{2}+\sigma}}\quad(z\in\mathbf{D}(\theta),s=0,1,\ldots),\tag{11·2}$$

(cf. (5·1) and (5·4)).

In order to establish bounds for the coefficients $A_s(\theta,z)$ and $B_s(\theta,z)$ analogous to (5·5), we introduce an auxiliary region $\mathbf{G}(\theta,\delta)$ defined as $\mathbf{G}(\theta)$ plus the aggregate of all points whose distance from a boundary point z_0 of $\mathbf{G}(\theta)$ is less than $\delta|z_0|^{-\frac{1}{2}}$. We restrict

$$0<\delta<\min(d,\tfrac{1}{2}b^{\frac{3}{2}}).\tag{11·3}$$

Clearly $\mathbf{G}(\theta,\delta)$ is contained in $\mathbf{D}(\theta)$, and if η is an arbitrary number such that $0<\eta<\delta$ then $\mathbf{G}(\theta,\delta)$ itself contains $\mathbf{G}(\theta,\eta)$.

As a preliminary we prove that *if* $z\in\mathbf{G}(\theta,\eta)$ *and* $|z|\geqslant b$, *then* $\mathbf{G}(\theta,\delta)$ *contains the circle*

$$|t-z|\leqslant\kappa(\delta-\eta)|z|^{-\frac{1}{2}},\tag{11·4}$$

where

$$\kappa=(1+db^{-\frac{3}{2}})^{-\frac{1}{2}}.\tag{11·5}$$

Suppose first that z is a point of $\mathbf{G}(\theta,\eta)$ not contained in $\mathbf{G}(\theta)$. There then exists z_0 on the boundary of $\mathbf{G}(\theta)$ such that $|z-z_0|<\eta|z_0|^{-\frac{1}{2}}$, and for points t satisfying (11·4) we have

$$|t-z_0|=|(t-z)+(z-z_0)|<\kappa(\delta-\eta)|z|^{-\frac{1}{2}}+\eta|z_0|^{-\frac{1}{2}}.\tag{11·6}$$

Now $$|z_0|=|(z_0-z)+z|<\eta|z_0|^{-\frac{1}{2}}+|z|<|z|(db^{-\frac{3}{2}}+1)=|z|\kappa^{-2}.$$

Thus $\kappa|z|^{-\frac{1}{2}}<|z_0|^{-\frac{1}{2}}$, and substituting this inequality in the right of (11·6) we see that $|t-z_0|<\delta|z_0|^{-\frac{1}{2}}$, and hence that $t\in\mathbf{G}(\theta,\delta)$.

Alternatively, suppose that $z\in\mathbf{G}(\theta)$. Then if t is a point of the circle (11·4) which is not itself contained in $\mathbf{G}(\theta)$, there must be at least one boundary point z_0 of $\mathbf{G}(\theta)$ on the join of t and z. In this event

$$|z_0|=|(z_0-z)+z|\leqslant|t-z|+|z|\leqslant\kappa(\delta-\eta)|z|^{-\frac{1}{2}}+|z|<|z|\{1+(\delta-\eta)|z|^{-\frac{3}{2}}\},$$

since $\kappa<1$. Therefore $|z_0|<|z|\kappa^{-2}$. From this result and (11·4) we obtain

$$|t-z_0|\leqslant|t-z|\leqslant\kappa(\delta-\eta)|z|^{-\frac{1}{2}}<(\delta-\eta)|z_0|^{-\frac{1}{2}}<\delta|z_0|^{-\frac{1}{2}}.$$

This completes the proof of the statement that $\mathbf{G}(\theta,\delta)$ contains the circle (11·4).

A further observation concerning points z of the region $\mathbf{G}(\theta,\delta)$ is that a path joining z and $c(\theta)$ exists lying in $\mathbf{G}(\theta,\delta)$ and having the property

$$\int_{c(\theta)}^{z}\frac{|\mathrm{d}t|}{1+|t|^{1+\sigma_1}}<k.\tag{11·7}$$

For if $z\in\mathbf{G}(\theta)$ we set $\int_{c(\theta)}^{z}=\int_0^z-\int_0^{c(\theta)}$, taking the paths of integration in the right-hand integrals to be those postulated by (9·14). If z is any other point of $\mathbf{G}(\theta,\delta)$, then z_0 exists on the boundary of $\mathbf{G}(\theta)$ such that $|z-z_0|<\delta|z_0|^{-\frac{1}{2}}$; we take the path in (11·7) to be the known path joining $c(\theta)$ and z_0 plus the join of z_0 and z; on the latter

$$\int_{z_0}^{z}\frac{|\mathrm{d}t|}{1+|t|^{1+\sigma_1}}<|z-z_0|<db^{-\frac{1}{2}}=k.$$

LEMMA

$$\left.\begin{aligned}&|A_s(\theta,z)|<k_s, && |A_s'(\theta,z)|<\frac{k_s}{1+|z|^{1+\sigma_1}}, && |A_s''(\theta,z)|<\frac{k_s}{1+|z|^{\frac{1}{2}+\sigma_1}},\\ &|B_s(\theta,z)|<\frac{k_s}{1+|z|^{\frac{1}{2}}}, && |B_s'(\theta,z)|<\frac{k_s}{1+|z|^{\frac{3}{2}}}, && |B_s''(\theta,z)|<\frac{k_s}{1+|z|^{1+\sigma_1}},\end{aligned}\right\} \tag{11·8}$$

when $z\in\mathbf{G}(\theta)$.

This corresponds to the lemma of § 5 and we prove it in a similar manner. Let us suppose temporarily that

$$|A_n'(\theta,z)|<\frac{k_n}{1+|z|^{1+\sigma_1}}\quad(z\in\mathbf{G}(\theta,\delta),n=0,1,\ldots,s). \tag{11·9}$$

Clearly
$$A_n(\theta,z)=\int_{c(\theta)}^{z}A_n'(\theta,t)\,\mathrm{d}t+A_n\{\theta,c(\theta)\}. \tag{11·10}$$

If $z\in\mathbf{G}(\theta,\delta)$ we may take the path of integration to be the same as that for the integral (11·7), and substituting (9·16) with $s=n$ and (11·9) in (11·10), we deduce that

$$|A_n(\theta,z)|<k_n\int_{c(\theta)}^{z}\frac{|\mathrm{d}t|}{1+|t|^{1+\sigma_1}}+k_n<k_n\quad(z\in\mathbf{G}(\theta,\delta)) \tag{11·11}$$

(cf. the first of (11·8)).

From Cauchy's integral formula we have

$$A_n''(\theta,z)=\frac{1}{2\pi\mathrm{i}}\int\frac{A_n'(\theta,t)}{(t-z)^2}\,\mathrm{d}t, \tag{11·12}$$

in which we may take the path of integration to be the circumference of the circle (11·4). If, now, $z\in\mathbf{G}(\theta,\eta)$ and $|z|\geqslant b$, then as we have seen above $t\in\mathbf{G}(\theta,\delta)$. Moreover,

$$|t|=|z+(t-z)|\geqslant|z|-\kappa(\delta-\eta)|z|^{-\frac{1}{2}}>|z|(1-\kappa\delta|z|^{-\frac{3}{2}})\geqslant|z|(1-\kappa\delta b^{-\frac{3}{2}})>\tfrac{1}{2}|z|, \tag{11·13}$$

since $\kappa<1$ and $\delta\leqslant\frac{1}{2}b^{\frac{3}{2}}$ (cf. (11·3) and (11·5)). Substituting (11·9) in (11·12) and using (11·13), we find that

$$|A_n''(\theta,z)|<\frac{1}{2\pi}\int\frac{k_n}{1+|t|^{1+\sigma_1}}\frac{|\mathrm{d}t|}{|t-z|^2}<\frac{k_n}{1+|\frac{1}{2}z|^{1+\sigma_1}}\frac{|z|^{\frac{1}{2}}}{\kappa(\delta-\eta)}<\frac{k_n}{1+|z|^{\frac{1}{2}+\sigma_1}} \tag{11·14}$$

(cf. the third of (11·8)). This result has been proved for $z\in\mathbf{G}(\theta,\eta)$ on the assumption that $|z|\geqslant b$. Application of the maximum-modulus theorem shows that it must also hold for $|z|<b$.

From (11·14), (11·11) and (11·2) we deduce immediately that

$$|-A_n''(\theta,z)+f_0(\theta,z)A_n(\theta,z)+\ldots+f_n(\theta,z)A_0(\theta,z)|<\frac{k_n}{1+|z|^{\frac{1}{2}+\sigma_1}}. \tag{11·15}$$

Now consider formula (9·12) for $B_s(\theta,z)$, with s replaced by n. If $|z|\leqslant b$ the path of integra- can be taken to be the join of z and the origin. Then clearly

$$|B_n(\theta,z)|<\tfrac{1}{2}|z|^{-\frac{1}{2}}\int_0^z|t|^{-\frac{1}{2}}\frac{k_n|\mathrm{d}t|}{1+|t|^{\frac{1}{2}+\sigma_1}}\leqslant\tfrac{1}{2}k_n|z|^{-\frac{1}{2}}\int_0^z|t^{-\frac{1}{2}}\mathrm{d}t|=k_n. \tag{11·16}$$

Alternatively, suppose that z is a point of $\mathbf{G}(\theta, \eta)$ outside the circle $|z| = b$. Then (11·7), with $c(\theta)$ replaced by 0, shows that a path exists joining 0 and z and lying in $\mathbf{G}(\theta, \eta)$, along which

$$\int_0^z \frac{|\mathrm{d}t|}{1+|t|^{1+\sigma_1}} < k.$$

This path, deformed if necessary so that the part inside $|z| = b$ is a straight line, is now taken as the path of integration in (9·12). Let z_1 be its meet with $|z| = b$. Then we have

$$|B_n(\theta, z)| < \tfrac{1}{2}k_n |z|^{-\frac{1}{2}} \left\{\int_0^{z_1} + \int_{z_1}^{z}\right\} \frac{|t^{-\frac{1}{2}}\,\mathrm{d}t|}{1+|t|^{\frac{1}{2}+\sigma_1}} < \frac{k_n b^{\frac{1}{2}}}{|z|^{\frac{1}{2}}} + \frac{k_n}{|z|^{\frac{1}{2}}} \int_{z_1}^{z} \frac{|\mathrm{d}t|}{1+|t|^{1+\sigma_1}} < \frac{k_n}{|z|^{\frac{1}{2}}}. \tag{11·17}$$

Combining (11·16) and (11·17) and again adjusting k_n, we find

$$|B_n(\theta, z)| < \frac{k_n}{1+|z|^{\frac{1}{2}}} \quad (z \in \mathbf{G}(\theta, \eta)) \tag{11·18}$$

(cf. the fourth of (11·8)).

Next, from (9·10) we have

$$2zB_n' = f_0 A_n + f_1 A_{n-1} + \ldots + f_n A_0 - A_n'' - B_n. \tag{11·19}$$

Substituting (11·2), (11·11), (11·14) and (11·18), we find that

$$|2zB_n'(\theta, z)| < \frac{k_n}{1+|z|^{\frac{1}{2}}} \quad (z \in \mathbf{G}(\theta, \eta)), \tag{11·20}$$

and with the aid of the maximum-modulus theorem we deduce that

$$|B_n'(\theta, z)| < \frac{k_n}{1+|z|^{\frac{3}{2}}} \quad (z \in \mathbf{G}(\theta, \eta)) \tag{11·21}$$

(cf. the fifth of (11·8)).

Again, differentiation of (11·19) yields

$$2zB_n'' = \frac{\mathrm{d}}{\mathrm{d}z}(f_0 A_n + f_1 A_{n-1} + \ldots + f_n A_0) - A_n''' - 3B_n'. \tag{11·22}$$

Since
$$|f_0 A_n + f_1 A_{n-1} + \ldots + f_n A_0| < \frac{k_n}{1+|z|^{\frac{1}{2}+\sigma_1}} \quad (z \in \mathbf{G}(\theta, \delta)), \tag{11·23}$$

we deduce by application of Cauchy's integral formula and the maximum-modulus theorem that

$$\left|\frac{\mathrm{d}}{\mathrm{d}z}(f_0 A_n + f_1 A_{n-1} + \ldots + f_n A_0)\right| < \frac{k_n}{1+|z|^{\sigma_1}} \quad (z \in \mathbf{G}(\theta, \eta)) \tag{11·24}$$

(cf. (11·9) and the analysis from (11·12) to (11·14)). Also, we have

$$A_n'''(\theta, z) = \frac{1}{\pi \mathrm{i}} \int_{|t-z| = \kappa(\delta-\eta)|z|^{-\frac{1}{2}}} \frac{A_n'(\theta, t)}{(t-z)^3}\,\mathrm{d}t. \tag{11·25}$$

Hence
$$|A_n'''(\theta, z)| < \frac{k_n}{1+|z|^{\sigma_1}} \quad (z \in \mathbf{G}(\theta, \eta)). \tag{11·26}$$

Substituting the last two inequalities and (11·21) in (11·22), we obtain

$$|2zB_n''(\theta, z)| < \frac{k_n}{1+|z|^{\sigma_1}} + \frac{k_n}{1+|z|^{\frac{3}{2}}} < \frac{k_n}{1+|z|^{\sigma_1}} \quad (z \in \mathbf{G}(\theta, \eta)), \tag{11·27}$$

since $\sigma_1 \leqslant \frac{3}{2}$ (cf. (9·15)). Hence

$$|B_n''(\theta, z)| < \frac{k_n}{1+|z|^{1+\sigma_1}} \quad (z \in \mathbf{G}(\theta, \eta)) \tag{11·28}$$

(cf. the sixth of (11·8)).

Finally, we substitute (11·2), (11·18) and (11·28) in (9·11), with s replaced by $s-1$, to obtain

$$|A_{s+1}'(\theta, z)| < \frac{k_{s+1}}{1+|z|^{1+\sigma_1}} \quad (z \in \mathbf{G}(\theta, \eta)). \tag{11·29}$$

The lemma is now established from the relations (11·9) and (11·29) by induction.

12. Equation satisfied by the truncated series

For brevity, we shall write $v \equiv u^{\frac{2}{3}}$. We define

$$L_m(u, \theta, z) = P_1(vz) \sum_{s=0}^{m} \frac{A_s(\theta, z)}{u^s} + \frac{P_1'(vz)}{v^2} \sum_{s=0}^{m-1} \frac{B_s(\theta, z)}{u^s} \tag{12·1}$$

(cf. (6·1)). Then we have

$$\frac{d}{dz} L_m(u, \theta, z) = P_1(vz) \left\{ \sum_{s=0}^{m-1} \frac{C_s(\theta, z)}{u^s} + \frac{A_m'(\theta, z)}{u^m} \right\} + vP_1'(vz) \left\{ \sum_{s=0}^{m} \frac{D_s(\theta, z)}{u^s} + \frac{B_{m-1}'(\theta, z)}{u^{m+1}} \right\} \tag{12·2}$$

(cf. (6·2)), and

$$\frac{d^2}{dz^2} L_m(u, \theta, z) - \{u^2 z + f(u, \theta, z)\} L_m(u, \theta, z) = R_m(u, \theta, z) \tag{12·3}$$

(cf. (6·4)), where $R_m(u, \theta, z)$ is regular in $\mathbf{D}(\theta)$ and is given by

$$R_m(u, \theta, z) = P_1(vz)\, R_m^{(1)}(u, \theta, z) + v^{-2} P_1'(vz)\, R_m^{(2)}(u, \theta, z), \tag{12·4}$$

$$R_m^{(1)}(u, \theta, z) = u^2 \left(\sum_{s=0}^{m} \frac{E_s}{u^s} + \frac{A_{m-1}'' + 2zB_{m-1}' + B_{m-1}}{u^{m+1}} + \frac{A_m''}{u^{m+2}} \right) - (u^2 z + f) \sum_{s=0}^{m} \frac{A_s}{u^s}, \tag{12·5}$$

$$R_m^{(2)}(u, \theta, z) = u^2 \left(\sum_{s=0}^{m-1} \frac{F_s}{u^s} + \frac{2A_m' + B_{m-2}''}{u^m} + \frac{B_{m-1}''}{u^{m+1}} \right) - (u^2 z + f) \sum_{s=0}^{m-1} \frac{B_s}{u^s}. \tag{12·6}$$

The inequality (9·2) implies that

$$f(u, \theta, z) = \sum_{s=0}^{m-1} \frac{f_s}{u^s} + \frac{\varpi k_m}{(1+|z|^{\frac{1}{2}+\sigma})\, u^m} \quad (z \in \mathbf{D}(\theta), u \geqslant u_0), \tag{12·7}$$

where $|\varpi| \leqslant 1$. Substituting (12·7) in (12·5) we find that the coefficients of $u^2, u, \ldots, u^{-m+1}$ all vanish as a consequence of (9·10) and the first of (9·9), and with the aid of (11·2) and (11·8) we may verify that

$$|R_m^{(1)}(u, \theta, z)| < \frac{1}{1+|z|^{\frac{1}{2}+\sigma_1}} \frac{k_m}{u^m} \quad (z \in \mathbf{G}(\theta), u \geqslant u_0). \tag{12·8}$$

Similarly

$$|R_m^{(2)}(u, \theta, z)| < \frac{1}{1+|z|^{1+\sigma_1}} \frac{k_m}{u^{m-1}} \quad (z \in \mathbf{G}(\theta), u \geqslant u_0). \tag{12·9}$$

Substituting (12·8) and (12·9) in (12·4) and using the inequalities

$$|P_1(z)| < \frac{k\,|\exp(-\frac{2}{3}z^{\frac{3}{2}})|}{1+|z|^{\frac{1}{4}}} \qquad |P_1'(z)| < k(1+|z|^{\frac{1}{4}})\,|\exp(-\tfrac{2}{3}z^{\frac{3}{2}})| \quad (|\arg z| \leqslant \pi), \tag{12·10}$$

(Olver 1954*a*, (4·5)), we deduce (cf. (6·7)) that

$$|R_m(u,\theta,z)| < |\exp(-\tfrac{2}{3}uz^{\frac{3}{2}})| \left\{\frac{1}{1+|vz|^{\frac{1}{4}}}\frac{1}{1+|z|^{\frac{1}{2}+\sigma_1}} + \frac{1}{v^{\frac{1}{2}}}\frac{1+|vz|^{\frac{1}{4}}}{1+|z|^{1+\sigma_1}}\right\}\frac{k_m}{u^m}$$
$$< \frac{|\exp(-\tfrac{2}{3}uz^{\frac{3}{2}})|}{1+|vz|^{\frac{1}{4}}}\frac{1}{1+|z|^{\frac{1}{2}+\sigma_1}}\frac{k_m}{u^m} \quad (z\in \mathbf{G}(\theta),\ u\geqslant u_0). \tag{12·11}$$

13. Proof of theorem B

A proof only for $j=1$ will be recorded. Exactly similar proofs can be formulated for the cases $j=2,3$; alternatively, the results for these cases can be deduced from that for $j=1$ by appropriate changes of the variables. We first suppose that $a_1 \equiv a_1(\theta)$ is a point of $\mathbf{G}(\theta)$ which is not at infinity.

Let us write

$$B_s^*(\theta,z) = (1+|z|^{\frac{1}{2}})\,B_s(\theta,z), \qquad C_s^*(\theta,z) = (1+|z|^{\frac{1}{2}})^{-1}C_s(\theta,z). \tag{13·1}$$

Setting $z=a_1$, we see from (11·8) and (9·7) that

$$|A_s(\theta,a_1)|<k_s, \qquad |B_s^*(\theta,a_1)|<k_s, \qquad |C_s^*(\theta,a_1)|<k_s, \qquad |D_s(\theta,a_1)|<k_s, \tag{13·2}$$

Therefore by the lemma of § 4 functions $A(u,\theta)$, $B^*(u,\theta)$, $C^*(u,\theta)$ and $D(u,\theta)$ exist having the asymptotic expansions

$$\left.\begin{aligned} A(u,\theta) &\sim \sum_{s=0}^{\infty}\frac{A_s(\theta,a_1)}{u^s}, & B^*(u,\theta) &\sim \sum_{s=0}^{\infty}\frac{B_s^*(\theta,a_1)}{u^s},\\ C^*(u,\theta) &\sim \sum_{s=0}^{\infty}\frac{C_s^*(\theta,a_1)}{u^s}, & D(u,\theta) &\sim \sum_{s=0}^{\infty}\frac{D_s(\theta,a_1)}{u^s}, \end{aligned}\right\} \tag{13·3}$$

as $u\to+\infty$, uniformly valid with respect to θ. We define $W_1(u,\theta,z)$ to be the solution of the differential equation (9·1) satisfying the conditions

$$\left.\begin{aligned} W_1(u,\theta,a_1) &= P_1(va_1)\,A(u,\theta) + \frac{P_1'(va_1)}{v^2}\frac{B^*(u,\theta)}{1+|a_1|^{\frac{1}{2}}},\\ \left[\frac{d}{dz}W_1(u,\theta,z)\right]_{z=a_1} &= P_1(va_1)\,(1+|a_1|^{\frac{1}{2}})\,C^*(u,\theta) + vP_1'(va_1)\,D(u,\theta), \end{aligned}\right\} \tag{13·4}$$

(cf. (7·3)). From (9·1) and (12·3) we have at once

$$\frac{d^2}{dz^2}\{W_1(z)-L_m(z)\} - \{u^2z+f(z)\}\{W_1(z)-L_m(z)\} = -R_m(z), \tag{13·5}$$

the arguments u and θ being suppressed.

Let z_1 be a point of the region common to $\mathbf{H}_1(\theta)$, defined in § 9, and the sector $\mathbf{S}_1+\mathbf{S}_2$, defined in Olver 1954*a*, § 4. Then as a consequence of condition (iv) given at the end of § 9, the path $\mathscr{P}$ joining z_1 and a_1 must also lie in $\mathbf{S}_1+\mathbf{S}_2$; in particular, $a_1\in \mathbf{S}_1+\mathbf{S}_2$. This is clear from figures 1 and 2 of Olver 1954*a*. Let ζ be a typical point of the path. By analogy with (7·5) we define the sequence of functions $h_{m,n}(u,\theta,z_1,\zeta)\equiv h_{m,n}(\zeta)$ by $h_{m,0}(\zeta)=0$ and

$$h_{m,n}(\zeta) = \frac{2\pi\, e^{\frac{1}{6}\pi i}}{v}\int_{a_1}^{\zeta}\{P_2(v\zeta)\,P_1(vt)-P_1(v\zeta)\,P_2(vt)\}\{f(t)\,h_{m,n-1}(t)-R_m(t)\}\,dt$$
$$+\alpha_m P_1(v\zeta)+\beta_m P_2(v\zeta) \quad (n\geqslant 1), \tag{13·6}$$

where the path of integration is the part of $\mathscr{P}$ between ζ and a_1, and α_m, β_m are determined by the conditions

$$h_{m,n}(a_1) = W_1(a_1) - L_m(a_1), \qquad h'_{m,n}(a_1) = W'_1(a_1) - L'_m(a_1). \tag{13·7}$$

Differentiation of (13·6) and use of the Wronksian relation

$$P_1(v\zeta)\, P'_2(v\zeta) - P'_1(v\zeta)\, P_2(v\zeta) = \mathrm{e}^{-\frac{1}{6}\pi \mathrm{i}}/(2\pi) \tag{13·8}$$

yield the differential equation

$$\frac{\mathrm{d}^2}{\mathrm{d}\zeta^2} h_{m,n}(\zeta) - u^2 \zeta h_{m,n}(\zeta) = f(\zeta)\, h_{m,n-1}(\zeta) - R_m(\zeta) \quad (n \geqslant 1). \tag{13·9}$$

From (13·6) and (13·7) it is seen that

$$\left.\begin{aligned} \alpha_m P_1(va_1) + \beta_m P_2(va_1) &= W_1(a_1) - L_m(a_1), \\ \alpha_m P'_1(va_1) + \beta_m P'_2(va_1) &= v^{-1}\{W'_1(a_1) - L'_m(a_1)\}. \end{aligned}\right\} \tag{13·10}$$

These equations show that α_m and β_m are independent of n. Solving with the aid of (13·8), we find

$$\left.\begin{aligned} \alpha_m &= 2\pi\, \mathrm{e}^{\frac{1}{6}\pi \mathrm{i}}[P'_2(va_1)\, \{W_1(a_1) - L_m(a_1)\} - v^{-1} P_2(va_1)\, \{W'_1(a_1) - L'_m(a_1)\}], \\ \beta_m &= 2\pi\, \mathrm{e}^{\frac{1}{6}\pi \mathrm{i}}[v^{-1} P_1(va_1)\, \{W'_1(a_1) - L'_m(a_1)\} - P'_1(va_1)\, \{W_1(a_1) - L_m(a_1)\}]. \end{aligned}\right\} \tag{13·11}$$

From (12·1), (12·2) and (13·4) we obtain

$$W_1(a_1) - L_m(a_1) = P_1(va_1)\left\{A(u,\theta) - \sum_{s=0}^{m} \frac{A_s(\theta,a_1)}{u^s}\right\} + \frac{P'_1(va_1)}{v^2}\left\{\frac{B^*(u,\theta)}{1+|a_1|^{\frac{1}{2}}} - \sum_{s=0}^{m-1} \frac{B_s(\theta,a_1)}{u^s}\right\},$$

$$W'_1(a_1) - L'_m(a_1) = P_1(va_1)\left\{(1+|a_1|^{\frac{1}{2}})\, C^*(u,\theta) - \sum_{s=0}^{m-1} \frac{C_s(\theta,a_1)}{u^s} - \frac{A'_m(\theta,a_1)}{u^m}\right\}$$
$$+ vP'_1(va_1)\left\{D(u,\theta) - \sum_{s=0}^{m} \frac{D_s(\theta,a_1)}{u^s} - \frac{B'_{m-1}(\theta,a_1)}{u^{m+1}}\right\}.$$

Using these equations, the inequalities (12·10), the expansions (13·3) and the lemma of § 11, we may verify without difficulty that

$$|W_1(a_1) - L_m(a_1)| < \frac{k_m}{u^{m+1}} \frac{|\exp(-\frac{2}{3}ua_1^{\frac{3}{2}})|}{1+|va_1|^{\frac{1}{4}}} + \frac{k_m}{u^m} \frac{1}{v^2} \frac{1+|va_1|^{\frac{1}{4}}}{1+|a_1|^{\frac{1}{2}}} |\exp(-\frac{2}{3}ua_1^{\frac{3}{2}})| < \frac{k_m}{u^{m+1}} \frac{|\exp(-\frac{2}{3}ua_1^{\frac{3}{2}})|}{1+|va_1|^{\frac{1}{4}}}, \tag{13·12}$$

and

$$|W'_1(a_1) - L'_m(a_1)| < \frac{k_m}{u^m} \frac{1+|a_1|^{\frac{1}{2}}}{1+|va_1|^{\frac{1}{4}}} |\exp(-\frac{2}{3}ua_1^{\frac{3}{2}})| + \frac{k_m}{u^{m+1}} v(1+|va_1|^{\frac{1}{4}})\, |\exp(-\frac{2}{3}ua_1^{\frac{3}{2}})|$$
$$< \frac{k_m}{u^m}(1+|va_1|^{\frac{1}{4}})\, |\exp(-\frac{2}{3}ua_1^{\frac{3}{2}})|. \tag{13·13}$$

Substituting (13·12) and (13·13) in (13·11), and using (12·10) and the corresponding inequalities

$$|P_2(z)| < k \frac{|\exp \frac{2}{3}z^{\frac{3}{2}}|}{1+|z|^{\frac{1}{4}}}, \qquad |P'_2(z)| < k(1+|z|^{\frac{1}{4}})\, |\exp \tfrac{2}{3}z^{\frac{3}{2}}|, \tag{13·14}$$

the region of validity of which includes $\mathbf{S}_1 + \mathbf{S}_2$, we find that

$$|\alpha_m| < k_m u^{-m-\frac{2}{3}}, \qquad |\beta_m| < k_m u^{-m-\frac{2}{3}}\, |\exp(-\tfrac{4}{3}ua_1^{\frac{3}{2}})|. \tag{13·15}$$

With the aid of (13·14), (12·10) and the relations

$$|\exp(-\tfrac{2}{3}u\zeta^{\frac{3}{2}})| \geqslant |\exp(-\tfrac{2}{3}ut^{\frac{3}{2}})| \geqslant |\exp(-\tfrac{2}{3}ua_1^{\frac{3}{2}})|, \tag{13·16}$$

obtained from condition (iv) on $\mathscr{P}$ given in § 9, we deduce from (13·15) that

$$|\alpha_m P_1(v\zeta)+\beta_m P_2(v\zeta)| < \frac{k_m}{u^{m+\frac{2}{3}}}\frac{|\exp(-\frac{2}{3}u\zeta^{\frac{3}{2}})|}{1+|v\zeta|^{\frac{1}{4}}} \tag{13·17}$$

(cf. (7·10)).

Next, we have from (13·6)

$$h_{m,1}(\zeta) = -\frac{2\pi e^{\frac{1}{6}\pi i}}{v}\int_{a_1}^{\zeta}\{P_2(v\zeta)P_1(vt)-P_1(v\zeta)P_2(vt)\}R_m(t)\,dt+\alpha_m P_1(v\zeta)+\beta_m P_2(v\zeta). \tag{13·18}$$

Using (12·10), (12·11), (13·14) and (13·16), we readily show that

$$|P_2(v\zeta)P_1(vt)R_m(t)| < \frac{k_m}{u^m}\frac{|\exp(\frac{2}{3}u\zeta^{\frac{3}{2}}-\frac{4}{3}ut^{\frac{3}{2}})|}{(1+|v\zeta|^{\frac{1}{4}})(1+|vt|^{\frac{1}{4}})^2}\frac{1}{1+|t|^{\frac{1}{2}+\sigma_1}} < \frac{k_m}{u^m}\frac{|\exp(-\frac{2}{3}u\zeta^{\frac{3}{2}})|}{(1+|v\zeta|^{\frac{1}{4}})(1+|t|^{1+\sigma_1})},$$

and

$$|P_1(v\zeta)P_2(vt)R_m(t)| < \frac{k_m}{u^m}\frac{|\exp(-\frac{2}{3}u\zeta^{\frac{3}{2}})|}{(1+|v\zeta|^{\frac{1}{4}})(1+|vt|^{\frac{1}{4}})^2}\frac{1}{1+|t|^{\frac{1}{2}+\sigma_1}} < \frac{k_m}{u^m}\frac{|\exp(-\frac{2}{3}u\zeta^{\frac{3}{2}})|}{(1+|v\zeta|^{\frac{1}{4}})(1+|t|^{1+\sigma_1})}.$$

Substitution of the last two results and (13·17) in (13·18) yields

$$|h_{m,1}(\zeta)| < \frac{k_m}{u^{m+\frac{2}{3}}}\frac{|\exp(-\frac{2}{3}u\zeta^{\frac{3}{2}})|}{1+|v\zeta|^{\frac{1}{4}}}\left|\int_{a_1}^{\zeta}\frac{|dt|}{1+|t|^{1+\sigma_1}}+\frac{k_m}{u^{m+\frac{2}{3}}}\frac{|\exp(-\frac{2}{3}u\zeta^{\frac{3}{2}})|}{1+|v\zeta|^{\frac{1}{4}}} < \frac{k}{v}\frac{k_m}{u^m}\frac{|\exp(-\frac{2}{3}u\zeta^{\frac{3}{2}})|}{1+|v\zeta|^{\frac{1}{4}}}, \tag{13·19}$$

as a consequence of (9·21).

From (13·6) we also obtain

$$h_{m,2}(\zeta)-h_{m,1}(\zeta) = \frac{2\pi e^{\frac{1}{6}\pi i}}{v}\int_{a_1}^{\zeta}\{P_2(v\zeta)P_1(vt)-P_1(v\zeta)P_2(vt)\}f(t)\,h_{m,1}(t)\,dt. \tag{13·20}$$

From (11·1) and (13·19) we can show that $|vf(t)\,h_{m,1}(t)|$ satisfies the inequality (12·11) which bounds $|R_m(t)|$. It follows immediately that

$$|h_{m,2}(\zeta)-h_{m,1}(\zeta)| < \frac{k^2}{v^2}\frac{k_m}{u^m}\frac{|\exp(-\frac{2}{3}u\zeta^{\frac{3}{2}})|}{1+|v\zeta|^{\frac{1}{4}}},$$

and continuing the argument by induction we obtain

$$|h_{m,n+1}(\zeta)-h_{m,n}(\zeta)| < \left(\frac{k}{v}\right)^{n+1}\frac{k_m}{u^m}\frac{|\exp(-\frac{2}{3}u\zeta^{\frac{3}{2}})|}{1+|v\zeta|^{\frac{1}{4}}} \quad (n=0,1,\ldots), \tag{13·21}$$

where k_m and k are independent of n.

Differentiation of (13·6) yields

$$h'_{m,1}(\zeta) = -2\pi e^{\frac{1}{6}\pi i}\int_{a_1}^{\zeta}\{P'_2(v\zeta)P_1(vt)-P'_1(v\zeta)P_2(vt)\}R_m(t)\,dt+\alpha_m vP'_1(v\zeta)+\beta_m vP'_2(v\zeta),$$

and $h'_{m,n+1}(\zeta)-h'_{m,n}(\zeta)$

$$= 2\pi e^{\frac{1}{6}\pi i}\int_{a_1}^{\zeta}\{P'_2(v\zeta)P_1(vt)-P'_1(v\zeta)P_2(vt)\}f(t)\{h_{m,n}(t)-h_{m,n-1}(t)\}\,dt \quad (n\geqslant 1).$$

Substituting by means of the inequalities (11·1), (12·10), (12·11), (13·14), (13·15), (13·21) and using (9·21) and (13·16), we readily show that

$$|h'_{m,n+1}(\zeta)-h'_{m,n}(\zeta)|<\left(\frac{k}{v}\right)^n\frac{k_m}{u^m}(1+|v\zeta|^{\frac{1}{4}})\,|\exp(-\tfrac{2}{3}u\zeta^{\frac{3}{2}})| \quad (n=0,1,\ldots). \tag{13·22}$$

The inequalities (13·21) and (13·22) are the key results for this case and correspond to (7·17) and (7·20), respectively, in case A. The series $\Sigma\{h_{m,n+1}(\zeta)-h_{m,n}(\zeta)\}$ and $\Sigma\{h'_{m,n+1}(\zeta)-h'_{m,n}(\zeta)\}$ clearly converge if $v>k$, and carrying out an analysis exactly similar to that between (7·20) and (7·31) we can prove that the sums of these series are $W_1(\zeta)-L_m(\zeta)$ and $W'_1(\zeta)-L'_m(\zeta)$, respectively. Setting $\zeta=z_1$ and taking $v>2k$ in (13·21) and (13·22), we obtain

$$|W_1(z_1)-L_m(z_1)|<X_m, \qquad |W'_1(z_1)-L'_m(z_1)|<Y_m, \tag{13·23}$$

where
$$X_m=\frac{k_m}{u^{m+\frac{2}{3}}}\frac{|\exp(-\frac{2}{3}uz_1^{\frac{3}{2}})|}{1+|vz_1|^{\frac{1}{4}}}, \qquad Y_m=\frac{k_m}{u^m}(1+|vz_1|^{\frac{1}{4}})\,|\exp(-\tfrac{2}{3}uz_1^{\frac{3}{2}})|, \tag{13·24}$$

(cf. (7·32)).

In order to express the error terms in the forms implied by the equations (9·18) and (9·19), we set

$$\left.\begin{aligned} X_m&=X_m^{(1)}.P_1(vz_1)+X_m^{(2)}.v^{-2}(1+|z_1|^{\frac{1}{2}})^{-1}P'_1(vz_1),\\ Y_m&=Y_m^{(1)}.P_1(vz_1)\,(1+|z_1|^{\frac{1}{2}})+Y_m^{(2)}.vP'_1(vz_1),\end{aligned}\right\} \tag{13·25}$$

where (cf. (13·8))

$$X_m^{(1)}=2\pi\,\mathrm{e}^{\frac{1}{6}\pi\mathrm{i}}P'_2(vz_1)\,X_m, \qquad X_m^{(2)}=-2\pi\,\mathrm{e}^{\frac{1}{6}\pi\mathrm{i}}v^2(1+|z_1|^{\frac{1}{2}})\,P_2(vz_1)\,X_m,$$
$$Y_m^{(1)}=2\pi\,\mathrm{e}^{\frac{1}{6}\pi\mathrm{i}}(1+|z_1|^{\frac{1}{2}})^{-1}P'_2(vz_1)\,Y_m, \qquad Y_m^{(2)}=-2\pi\,\mathrm{e}^{\frac{1}{6}\pi\mathrm{i}}v^{-1}P_2(vz_1)\,Y_m.$$

With the aid of (13·14) and (13·24) we may show that

$$\left.\begin{aligned} |X_m^{(1)}|&<k_mu^{-m-\frac{2}{3}}, \qquad |X_m^{(2)}|<k_mu^{-m+\frac{2}{3}},\\ |Y_m^{(1)}|&<k_mu^{-m+\frac{1}{3}}, \qquad |Y_m^{(2)}|<k_mu^{-m-\frac{2}{3}}.\end{aligned}\right\} \tag{13·26}$$

The results (9·18) and (9·19) (with $z=z_1$) now follow on changing m into $m+1$ and using the lemma of § 11. In an exactly similar manner, or by an appeal to symmetry, we can establish the same results when z_1 is a point of the region common to $\mathbf{H}_1(\theta)$ and $\mathbf{S}_1+\mathbf{S}_3$. This completes the proof of theorem B for finite $a_1(\theta)$.

When $a_1(\theta)$ is at infinity the proof needs modification. The principal change is that $W_1(u,\theta,z)$ is now defined to be the solution of (9·1) with the conditions

$$\lim\{2\pi^{\frac{1}{2}}(vz)^{\frac{1}{4}}\exp(\tfrac{2}{3}uz^{\frac{3}{2}})\,W_1(u,\theta,z)\}=A(u,\theta)-u^{-1}\exp(\tfrac{1}{2}\mathrm{i}\arg a_1)\,B^*(u,\theta),$$
$$\lim\{2\pi^{\frac{1}{2}}(vz)^{-\frac{1}{4}}\exp(\tfrac{2}{3}uz^{\frac{3}{2}})\,(\mathrm{d}/\mathrm{d}z)\,W_1(u,\theta,z)\}=-v\{A(u,\theta)-u^{-1}\exp(\tfrac{1}{2}\mathrm{i}\arg a_1)\,B^*(u,\theta)\},$$

as $z\to a_1(\theta)$ along $\mathscr{L}$ (cf. (8·3)). The sequence $h_{m,n}(\zeta)$ is defined by (13·6) with $\beta_m=0$ and

$$\alpha_m=A(u,\theta)-\sum_{s=0}^{m}\frac{A_s(\theta,a_1)}{u^s}-\frac{\exp(\frac{1}{2}\mathrm{i}\arg a_1)}{u}\left\{B^*(u,\theta)-\sum_{s=0}^{m-1}\frac{B_s^*(\theta,a_1)}{u^s}\right\}$$

(cf. (8·5)). Other changes are similar to those given in § 8 for case A and do not need to be recorded here.

PART 3. CASE D

14. Statement of conditions and theorem D

The standard form of differential equation for this case is taken to be

$$\frac{d^2w}{dz^2} = \frac{1}{z}\frac{dw}{dz} + \left\{u^2 + \frac{\mu^2-1}{z^2} + f(u,\theta,\mu,z)\right\} w \tag{14·1}$$

(cf. (1·10)), in which u is a large positive parameter, θ is a set of real or complex parameters ranging over a set of values $\mathbf{\Theta}$, and μ is a real or complex parameter which ranges over a bounded region $\mathbf{M}$ lying in the half-plane $\operatorname{Re}\mu \geq 0$. We suppose that the variable z ranges over a simply-connected complex domain $\mathbf{D}(\theta,\mu)$, bounded or otherwise. Further, $\mathbf{D}(\theta,\mu)$ must contain the circle $|z| \leq b$, where b is positive and *fixed*, i.e. independent of u, θ and μ.

We assume that $f(u,\theta,\mu,z)$ is a regular *even* function of z in $\mathbf{D}(\theta,\mu)$ and that

$$\left| f(u,\theta,\mu,z) - \sum_{s=0}^{m-1} \frac{f_s(\theta,\mu,z)}{u^s} \right| < \frac{1}{1+|z|^{1+\sigma}} \frac{k_m}{u^m} \quad (m = 0,1,2,\ldots), \tag{14·2}$$

valid when $z \in \mathbf{D}(\theta,\mu)$, $\mu \in \mathbf{M}$, $\theta \in \mathbf{\Theta}$ and $u \geq u_0$, where u_0 is fixed. Here σ is fixed and positive, and each $f_s(\theta,\mu,z)$ is a regular even function of z in $\mathbf{D}(\theta,\mu)$. As in parts 1 and 2 the symbols k and k_m are used generically to denote numbers independent of u, θ, z and also, in this case, μ.

Following the analysis given in Olver 1956, § 7, we readily verify that formal solutions of (14·1) are given by

$$w = z\mathscr{Z}_\mu(uz) \sum_{s=0}^{\infty} \frac{A_s(\theta,\mu,z)}{u^s} + \frac{z}{u}\mathscr{Z}_{\mu+1}(uz) \sum_{s=0}^{\infty} \frac{B_s(\theta,\mu,z)}{u^s}, \tag{14·3}$$

$$\frac{dw}{dz} = \mathscr{Z}_\mu(uz) \sum_{s=0}^{\infty} \frac{C_s(\theta,\mu,z)}{u^s} + uz\mathscr{Z}_{\mu+1}(uz) \sum_{s=0}^{\infty} \frac{D_s(\theta,\mu,z)}{u^s}, \tag{14·4}$$

where $\mathscr{Z}_\mu$ denotes the modified cylinder function of order μ. Here A_0 = constant, A_1 = constant, and the higher coefficients are given by

$$2B_s = -A_s' + \int_0^z \left(f_0 A_s + f_1 A_{s-1} + \ldots + f_s A_0 - \frac{2\mu+1}{z} A_s' \right) dz, \tag{14·5}$$

$$2A_{s+2} = \frac{2\mu+1}{z} B_s - B_s' + \int (f_0 B_s + f_1 B_{s-1} + \ldots + f_s B_0)\, dz + \text{constant}. \tag{14·6}$$

In these equations the arguments θ, μ, z of A_s, B_s and f_s have been suppressed and primes used to denote differentiations with respect to z. These two equations determine a set of functions which are regular functions of z in $\mathbf{D}(\theta,\mu)$, A_s being even and B_s being odd. If the f_s of odd suffix vanish we can arrange that the same is true of both A_s and B_s. The coefficients C_s and D_s appearing in (14·4) are given by

$$C_s = (\mu+1) A_s + zA_s' + zB_s, \qquad D_s = A_s - \mu z^{-1} B_{s-2} + B_{s-2}', \tag{14·7}$$

and are regular even functions of z in $\mathbf{D}(\theta,\mu)$.

We prescribe $\mathbf{G}(\theta,\mu)$ to be a closed subdomain of $\mathbf{D}(\theta,\mu)$ which has the properties:

(i) $\mathbf{G}(\theta,\mu)$ contains the circle $|z| \leq b$.

(ii) The distance between each point of $\mathbf{G}(\theta,\mu)$ and each boundary point of $\mathbf{D}(\theta,\mu)$ has a positive lower bound which is independent of θ and μ.

(iii) $$\int_0^z \frac{|\mathrm{d}t|}{1+|t|^{1+\sigma_1}} < k \quad (z \in \mathbf{G}(\theta,\mu)), \tag{14·8}$$

for some path lying wholly in $\mathbf{G}(\theta,\mu)$, where

$$\sigma_1 = \min(\sigma, 1). \tag{14·9}$$

The arbitrary constants occurring in (14·6) are subjected to the condition

$$|A_s\{\theta,\mu,c(\theta,\mu)\}| < k_s, \tag{14·10}$$

where $c(\theta,\mu)$ is a prescribed point of $\mathbf{G}(\theta,\mu)$. The asymptotic nature of the formal series (14·3) and (14·4) can now be expressed in the following theorem:

THEOREM D. *The differential equation* (14·1) *possesses solutions* $W_j(u,\theta,\mu,z)$ $(j = 1, 2)$ *with the properties*

$$W_1(u,\theta,\mu,z) = zI_\mu(uz)\left\{\sum_{s=0}^{m}\frac{A_s}{u^s}+O\left(\frac{1}{u^{m+1}}\right)\right\}+\frac{z}{u}I_{\mu+1}(uz)\left\{\sum_{s=0}^{m-1}\frac{B_s}{u^s}+\frac{z}{1+|z|}O\left(\frac{1}{u^m}\right)\right\}, \tag{14·11}$$

$$\frac{\mathrm{d}}{\mathrm{d}z}W_1(u,\theta,\mu,z) = I_\mu(uz)\left\{\sum_{s=0}^{m-1}\frac{C_s}{u^s}+(1+|z|)\,O\left(\frac{1}{u^m}\right)\right\}+uzI_{\mu+1}(uz)\left\{\sum_{s=0}^{m}\frac{D_s}{u^s}+O\left(\frac{1}{u^{m+1}}\right)\right\}, \tag{14·12}$$

$$W_2(u,\theta,\mu,z) = zK_\mu(uz)\left\{\sum_{s=0}^{m}\frac{A_s}{u^s}+O\left(\frac{1}{u^{m+1}}\right)\right\}-\frac{z}{u}K_{\mu+1}(uz)\left\{\sum_{s=0}^{m-1}\frac{B_s}{u^s}+\frac{z}{1+|z|}O\left(\frac{1}{u^m}\right)\right\}, \tag{14·13}$$

$$\frac{\mathrm{d}}{\mathrm{d}z}W_2(u,\theta,\mu,z) = K_\mu(uz)\left\{\sum_{s=0}^{m-1}\frac{C_s}{u^s}+(1+|z|)\,O\left(\frac{1}{u^m}\right)\right\}-uzK_{\mu+1}(uz)\left\{\sum_{s=0}^{m}\frac{D_s}{u^s}+O\left(\frac{1}{u^{m+1}}\right)\right\}, \tag{14·14}$$

as $u \to +\infty$, *valid when* $\theta \in \mathbf{\Theta}$, $\mu \in \mathbf{M}$ *and* $z \in \mathbf{H}_j(\theta,\mu)$. *Here m denotes an arbitrary positive integer or zero; the solutions are independent of m. Each of the O's is uniform with respect to* θ, μ *and z, except that in the case* $j = 2$ *the part of* $\mathbf{M}$ *common to the annulus* $0 < |\mu| < \delta$ *is excluded,* δ *being an arbitrary positive number.*

The regions of validity $\mathbf{H}_j(\theta,\mu)$ and the branches of the modified Bessel functions are defined as follows.

$\mathbf{H}_1(\theta,\mu)$ comprises those points z of $\mathbf{G}(\theta,\mu)$ which can be joined to the origin by a path $\mathscr{P}$ having the following properties, t being a typical point of $\mathscr{P}$.

(i) $\mathscr{P}$ lies in $\mathbf{G}(\theta,\mu)$.

(ii) $\mathscr{P}$ comprises a finite number of Jordan arcs, each with parametric equation of the form $t = t(\tau)$, where τ is the real parameter of the arc; $t''(\tau)$ is continuous and $t'(\tau)$ does not vanish.

(iii) $$\int_{\mathscr{P}} \frac{|\mathrm{d}t|}{1+|t|^{1+\sigma_1}} < k. \tag{14·15}$$

(iv) Re t is monotonic as t traverses $\mathscr{P}$ from 0 to z.

(v) If t, ζ are any two points of $\mathscr{P}$ arranged in the order 0, t, ζ, z along $\mathscr{P}$, then

$$|t| < k|\zeta|. \tag{14·16}$$

Equations (14·11) and (14·12) then hold for $z \in \mathbf{H}_1(\theta,\mu)$ uniformly with respect to *all* values of arg z.

In the definition of $\mathbf{H}_2(\theta,\mu)$ we suppose $a(\theta,\mu)$ to be any prescribed point of $\mathbf{G}(\theta,\mu)$ which is outside or on the circle $|z| = b$, or the point at infinity on a straight line $\mathscr{L}$ lying in $\mathbf{G}(\theta,\mu)$. In either event we impose the restriction

$$|\arg a(\theta,\mu)| < \tfrac{1}{2}\pi. \tag{14·17}$$

Then $\mathbf{H}_2(\theta, \mu)$ comprises those points z of $\mathbf{G}(\theta, \mu)$ which can be joined to $a(\theta, \mu)$ by a path $\mathscr{P}$ having the properties (i), (ii), (iii) listed above for the case $j = 1$; in addition $\mathscr{P}$ must coincide with $\mathscr{L}$ for all sufficiently large $|t|$, should $a(\theta, \mu)$ be at infinity. In place of (iv) and (v) we have:

(iv) Re t is monotonic decreasing as t traverses $\mathscr{P}$ from $a(\theta, \mu)$ to z.

(v) If t, ζ are any two points of $\mathscr{P}$ arranged in the order a, t, ζ, z along $\mathscr{P}$, then

$$|\zeta| < k|t|. \tag{14·18}$$

If Re $z \geqslant 0$ the branches of the modified Bessel functions $K_\mu(uz)$ and $K_{\mu+1}(uz)$ in (14·13) and (14·14) are the principal ones. If Re $z < 0$ the branches are determined by $\frac{1}{2}\pi < \arg z < \frac{3}{2}\pi$ or $-\frac{1}{2}\pi > \arg z > -\frac{3}{2}\pi$ according as $\mathscr{P}$ intersects the positive or negative imaginary axis.

15. Remarks on theorem D

Remarks (i) and (ii) made in § 3 in connexion with case A apply equally well here to the conditions (14·8) and (14·15) provided, of course, that we stipulate uniformity with respect to μ. Corresponding to § 3 (iii) we have:

(iii) If the inequality (14·2) holds only for $m \leqslant M$, where M is a fixed positive integer or zero, then equations (14·11) to (14·14) hold for $m \leqslant M-1$. They also hold for $m = M$ provided that the error terms $O(1/u^m)$ appearing in (14·11) and (14·13) are changed to $O(1/u^{M-1})$.

We also make the following observations:

(iv) For $j = 2$, condition (v) implies that $\mathscr{P}$ cannot pass through the origin; this is seen by taking $t = 0$ in (14·18). From this result and condition (iv) we see that $\mathscr{P}$ cannot intersect both the positive and the negative imaginary axes. Thus there is no ambiguity in the choice of branch of $K_\mu(uz)$ and $K_{\mu+1}(uz)$. If, however, for a given point z(Re $z < 0$) two alternative paths $\mathscr{P}$ exist which satisfy conditions (i) to (v) and intersect the imaginary axis on opposite sides of the origin, then equations (14·13) and (14·14) hold for both $\frac{1}{2}\pi < \arg z < \frac{3}{2}\pi$ and $-\frac{1}{2}\pi > \arg z > -\frac{3}{2}\pi$.

(v) An investigation has not been made of the necessity of the condition that the annulus $0 < |\mu| < \delta$ be excluded from the μ-region of *uniform* validity when $j = 2$. This would appear to require the use of a sharper inequality for the Bessel function $K_\mu(z)$ than (16·2) given below. We may note however that since $\mu = 0$ is not excluded we can show, by proper choice of δ, that equations (14·13) and (14·14) hold for any *fixed* value of μ. Alternative forms of error terms which hold uniformly throughout $\mathbf{M}$, including the annulus, are given by equations (19·14) and (19·15).

16. Bounds for the Bessel functions

We require bounds for the modified Bessel functions $I_\mu(z)$ and $K_\mu(z)$ which are uniformly valid with respect to bounded μ in the half-plane Re $\mu \geqslant 0$. These are afforded by the following results which are extensions of the inequalities for fixed μ given in Olver 1956, § 9:

$$|I_\mu(z)| < kV_\mu(z) \quad (\mu \in \mathbf{M}, |\arg z| \leqslant \tfrac{1}{2}\pi), \tag{16·1}$$

$$|K_\mu(z)| < kX_\mu(z) \quad (\mu \in \mathbf{M}, |\arg z| \leqslant \tfrac{3}{2}\pi), \tag{16·2}$$

Here
$$V_\mu(z) = \frac{|z^\alpha e^z|}{1+|z|^{\alpha+\frac{1}{2}}}, \qquad X_\mu(z) = l_\mu(z)\,\frac{1+|z|^\alpha}{1+|z|^{\frac{1}{2}}}\frac{|e^{-z}|}{|z|^\alpha}, \tag{16·3}$$

$$l_\mu(z) = \ln\frac{1+2|z|}{|z|} \quad (|\mu|<\delta), \qquad l_\mu(z) = 1 \quad (|\mu|\geqslant\delta), \tag{16·4}$$

where
$$\alpha = \operatorname{Re}\mu\,(\geqslant 0), \tag{16·5}$$

and δ is an arbitrary positive number in the range $0<\delta<\frac{1}{2}$. In (16·1) and (16·2), k denotes a generic number independent of μ and z; the k of (16·2) depends on δ.

These results may be proved as follows.

Large z. An examination of the analysis given in Watson 1944, § 7·2 reveals that the asymptotic expansions of Hankel for the Hankel functions $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$ when $|z|$ is large, are uniformly valid with respect to ν and $\arg z$ when $\nu\in\mathbf{M}$ and

$$-\pi+\epsilon\leqslant \arg z\leqslant 2\pi-\epsilon \quad (\text{in the case of } H_\nu^{(1)}(z)),$$
$$-2\pi+\epsilon\leqslant \arg z\leqslant \pi-\epsilon \quad (\text{in the case of } H_\nu^{(2)}(z)),$$

where ϵ is an arbitrary fixed positive number.

Hence we derive (Watson 1944, § 7·23)

$$I_\mu(z) = \frac{e^z}{(2\pi z)^{\frac{1}{2}}}O(1) + \frac{e^{-z\pm(\mu+\frac{1}{2})\pi i}}{(2\pi z)^{\frac{1}{2}}}O(1) \quad \begin{Bmatrix} -\frac{1}{2}\pi+\epsilon\leqslant\arg z\leqslant\frac{3}{2}\pi-\epsilon \text{ (upper sign)} \\ -\frac{3}{2}\pi+\epsilon\leqslant\arg z\leqslant\frac{1}{2}\pi-\epsilon \text{ (lower sign)} \end{Bmatrix}, \tag{16·6}$$

and
$$K_\mu(z) = \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z}O(1) \quad (|\arg z|\leqslant\tfrac{3}{2}\pi-\epsilon), \tag{16·7}$$

as $|z|\to\infty$, where the O's are uniform with respect to μ and $\arg z$.

Equation (16·6) enables us to prove (16·1) when $|z|>\kappa$, where $\kappa(>0)$ is assignable independently of μ and $\arg z$. Similarly, (16·7) establishes (16·2) when $|z|>\kappa$ and $|\arg z|\leqslant\frac{3}{2}\pi-\epsilon$; the extension to $|\arg z|\leqslant\frac{3}{2}\pi$ is achieved with the aid of the formula

$$K_\mu(z) = e^{\pm\mu\pi i}K_\mu(z\,e^{\pm\pi i}) \pm \pi i I_\mu(z\,e^{\pm\pi i}). \tag{16·8}$$

Bounded z. When $|z|\leqslant\kappa$ we use the series definitions

$$I_\mu(z) = \frac{(\frac{1}{2}z)^\mu}{\Gamma(1+\mu)}\left[1+\frac{(\frac{1}{2}z)^2}{1!(1+\mu)}+\frac{(\frac{1}{2}z)^4}{2!(1+\mu)(2+\mu)}+\dots\right], \tag{16·9}$$

$$I_{-\mu}(z) = \frac{(\frac{1}{2}z)^{-\mu}}{\Gamma(1-\mu)}\left[1+\frac{(\frac{1}{2}z)^2}{1!(1-\mu)}+\frac{(\frac{1}{2}z)^4}{2!(1-\mu)(2-\mu)}+\dots\right], \tag{16·10}$$

and
$$K_\mu(z) = \tfrac{1}{2}\pi\operatorname{cosec}\mu\pi\{I_{-\mu}(z)-I_\mu(z)\}, \tag{16·11}$$

the limit being taken in (16·11) when μ is an integer.

Since $\operatorname{Re}\mu\geqslant 0$ it follows that $|s+\mu|\geqslant s$ $(s = 1, 2, \dots)$, and from (16·9) we obtain

$$|I_\mu(z)|\leqslant\left|\frac{(\frac{1}{2}z)^\mu}{\Gamma(1+\mu)}\right|\left[1+\frac{(\frac{1}{2}\kappa)^2}{(1!)^2}+\frac{(\frac{1}{2}\kappa)^4}{(2!)^2}+\dots\right]<k|z|^\alpha, \tag{16·12}$$

when $|\arg z|$ is bounded. This enables us to complete the proof of (16·1).

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The proof of (16·2) is more complicated and we divide the region **M** into three parts.

(i) Suppose first that μ lies on or outside the circles

$$|\mu-n| = \delta \quad (n = 0, 1, 2, \ldots). \tag{16·13}$$

Only a finite number of these circles lie in **M**, and since $\delta<\frac{1}{2}$ they do not overlap.

Clearly $|s-\mu|\geqslant\delta$ for all positive integer values of s. Hence, from (16·10) we obtain

$$|I_{-\mu}(z)|<\left|\frac{(\frac{1}{2}z)^{-\mu}}{\Gamma(1-\mu)}\right|\left[1+\frac{(\frac{1}{2}\kappa)^2}{1!\,\delta}+\frac{(\frac{1}{2}\kappa)^4}{2!\,\delta^2}+\ldots\right]<k|z|^{-\alpha},$$

since $|\arg z|$ is bounded. Using this result, (16·12) and the fact that $|\operatorname{cosec}\mu\pi|$ is bounded here, we see from (16·11) that

$$|K_\mu(z)|<k|z|^{-\alpha}. \tag{16·14}$$

(ii) Next suppose that, for some positive integer n,

$$|\mu-n|\leqslant\delta. \tag{16·15}$$

For fixed z, other than zero, the function $z^\mu K_\mu(z)$ is a regular function of μ inside the circle (16·15), whilst from (16·14) we see that its modulus is bounded (independently of z) on the circumference. Hence by the maximum-modulus theorem

$$|K_\mu(z)|<k|z^{-\mu}|<k|z|^{-\alpha} \tag{16·16}$$

throughout the circle.

(iii) Finally, suppose that μ lies in the semi-circle

$$|\mu|\leqslant\delta, \quad \operatorname{Re}\mu\geqslant 0. \tag{16·17}$$

We write

$$K_\mu(z) = \tfrac{1}{2}\pi\{z^{-\mu}K^{(1)}_\mu(z)+z^\mu K^{(2)}_\mu(z)+K^{(3)}_\mu(z)\}, \tag{16·18}$$

where

$$K^{(1)}_\mu(z) = \frac{z^\mu I_{-\mu}(z)-I_0(z)}{\sin\mu\pi}, \qquad K^{(2)}_\mu(z) = \frac{I_0(z)-z^{-\mu}I_\mu(z)}{\sin\mu\pi}, \qquad K^{(3)}_\mu(z) = \frac{z^{-\mu}-z^\mu}{\sin\mu\pi}I_0(z).$$

Clearly $K^{(1)}_\mu(z)$ and $K^{(2)}_\mu(z)$ are regular functions of μ inside the circle $|\mu|\leqslant\delta$, whilst their moduli are bounded on the circumference. Applying the maximum-modulus theorem again, we deduce that

$$|K^{(1)}_\mu(z)|,\ |K^{(2)}_\mu(z)|<k \tag{16·19}$$

throughout the circle.

For the remaining function we write

$$K^{(3)}_\mu(z) = \ln z\, I_0(z)\frac{\mu}{\sin\mu\pi}\frac{e^{-x}-e^x}{x}, \tag{16·20}$$

where $x = \mu\ln z$. Clearly

$$\left|\frac{e^{-x}-e^x}{x}\right|<k\,e^{|\operatorname{Re}x|}<k|z|^{-\alpha}, \tag{16·21}$$

when μ lies in the semi-circle (16·17). Hence we have

$$|K^{(3)}_\mu(z)|<k|\ln z||z|^{-\alpha}<k\,l(z)|z|^{-\alpha}, \tag{16·22}$$

where

$$l(z) = \ln\frac{1+2|z|}{|z|} \tag{16·23}$$

(cf. (16·4)), and is a decreasing function of $|z|$. From this result and (16·18), (16·19), we obtain

$$|K_\mu(z)|<k|z|^{-\alpha}+k|z|^\alpha+k\,l(z)|z|^{-\alpha}<k\,l(z)|z|^{-\alpha}. \tag{16·24}$$

All regions of **M** have now been covered. Combining (16·14), (16·16) and (16·24), we see that

$$|K_\mu(z)| < k\, l_\mu(z)\, |z|^{-\alpha} \quad (\mu \in \mathbf{M}, |z| \leqslant \kappa, |\arg z| \leqslant \tfrac{3}{2}\pi), \tag{16·25}$$

where $l_\mu(z)$ is defined by (16·4). This completes the proof of (16·2).

Inequalities for products. Using (16·1) and (16·2) we can prove that the inequalities (9·12) given in Olver 1956 are uniformly valid with respect to $\mu \in \mathbf{M}$, provided that $l_\mu(z)$ is defined as in the present paper.

17. Preliminaries in the proof of theorem D

The proof of theorem D follows the pattern of the proofs of theorems A and B given in parts 1 and 2, and we shall record only the principal steps.

(i) *Bounds for the coefficients*

Lemma

$$\left.\begin{aligned} &|A_s(\theta,\mu,z)| < k_s, \quad &|A_s'(\theta,\mu,z)| < \frac{k_s}{1+|z|^{1+\sigma_1}}, \quad &|A_s''(\theta,\mu,z)| < \frac{k_s}{1+|z|^{1+\sigma_1}},\\ &|B_s(\theta,\mu,z)| < k_s, \quad &|B_s'(\theta,\mu,z)| < \frac{k_s}{1+|z|^{1+\sigma_1}}, \quad &|B_s''(\theta,\mu,z)| < \frac{k_s}{1+|z|^{1+\sigma_1}},\end{aligned}\right\} \tag{17·1}$$

when $z \in \mathbf{G}(\theta, \mu)$.

This is the extended form of lemma 2 given in Olver 1956, and may be proved by the same method (cf. also §§ 5 and 11 of the present paper).

(ii) *Equation satisfied by the truncated series*

Let $L_m(u,\theta,\mu,z) \equiv L_m(z)$ be defined by the equation

$$L_m(z) = z\mathscr{Z}_\mu(uz)\sum_{s=0}^{m}\frac{A_s}{u^s} + \frac{z}{u}\mathscr{Z}_{\mu+1}(uz)\sum_{s=0}^{m-1}\frac{B_s}{u^s}, \tag{17·2}$$

in which the arguments θ, μ, z of A_s and B_s have been suppressed. Then

$$L_m'(z) = \mathscr{Z}_\mu(uz)\left\{\sum_{s=0}^{m-1}\frac{C_s}{u^s} + \frac{(\mu+1)A_m + zA_m'}{u^m}\right\} + uz\mathscr{Z}_{\mu+1}(uz)\left\{\sum_{s=0}^{m}\frac{D_s}{u^s} + \frac{B_{m-1}' - \mu z^{-1}B_{m-1}}{u^{m+1}}\right\}, \tag{17·3}$$

and by analogy with §§ 6 and 12, $L_m(z)$ satisfies a differential equation of the form

$$L_m''(z) - \frac{1}{z}L_m'(z) - \left\{u^2 + \frac{\mu^2-1}{z^2} + f(z)\right\}L_m(z) = R_m(z), \tag{17·4}$$

where $f(z) \equiv f(u,\theta,\mu,z)$. The function $R_m(z) \equiv R_m(u,\theta,\mu,z)$ can be expressed in the form

$$R_m(z) = z\mathscr{Z}_\mu(uz)\, R_m^{(1)}(z) + z\mathscr{Z}_{\mu+1}(uz)\, R_m^{(2)}(z) \tag{17·5}$$

(cf. (12·4)), where $R_m^{(1)}(z) \equiv R_m^{(1)}(u,\theta,\mu,z)$ and $R_m^{(2)}(z) \equiv R_m^{(2)}(u,\theta,\mu,z)$ are regular functions of z in $\mathbf{D}(\theta,\mu)$, satisfying

$$|R_m^{(1)}(z)|, |R_m^{(2)}(z)| < \frac{1}{1+|z|^{1+\sigma_1}}\frac{k_m}{u^m} \quad (z \in \mathbf{G}(\theta,\mu), u \geqslant u_0). \tag{17·6}$$

18. Proof for the solution W_1

We define $W_1(u,\theta,\mu,z)\equiv W_1(z)$ to be the solution of the differential equation (14·1) satisfying the conditions

$$\lim\{z^{-\mu-1}W_1(z)\} = \frac{(\frac{1}{2}u)^\mu}{\Gamma(1+\mu)}A(u,\theta,\mu), \tag{18·1}$$

$$\lim\{z^{-\mu}W_1'(z)\} = (1+\mu)\frac{(\frac{1}{2}u)^\mu}{\Gamma(1+\mu)}A(u,\theta,\mu), \tag{18·2}$$

as $z\to 0$ along *any* straight line, where $A(u,\theta,\mu)$ is a function having the asymptotic expansion

$$A(u,\theta,\mu) \sim \sum_{s=0}^{\infty}\frac{A_s(\theta,\mu,0)}{u^s} \tag{18·3}$$

as $u\to+\infty$, uniformly valid with respect to θ and μ (cf. § 4).

For a given $A(u,\theta,\mu)$ the solution $W_1(z)$ exists; moreover, no other solution of (14·1) can exist with the properties (18·1) and (18·2) as $z\to 0$ along a *given* straight line. These statements can be proved by considering the forms of a fundamental pair of solutions $w_1(z)$, $w_2(z)$ of (14·1) in the neighbourhood of the origin, given by

$$\left.\begin{aligned} w_1(z) &= z^{1+\mu}\phi_\mu(z),\\ w_2(z) &= z^{1-\mu}\psi_\mu(z) \quad (2\mu\neq 0,1,2,\ldots),\\ w_2(z) &= z^{1-\mu}\psi_\mu(z)+z^{1+\mu}\ln z\,\chi_\mu(z) \quad (2\mu=0,1,2,\ldots),\end{aligned}\right\}$$

where $\phi_\mu(z)$, $\psi_\mu(z)$ and $\chi_\mu(z)$ are regular functions of z with the properties

$$\phi_\mu(0)\neq 0, \qquad \psi_\mu(0)\neq 0 \quad (\mu\neq 0), \qquad \chi_0(0)\neq 0,$$

(cf. Copson 1944, §§ 10·12 and 10·15). Since $\operatorname{Re}\mu\geqslant 0$ it is clear that $z^{-\mu-1}w_2(z)$ cannot tend to a finite limit unless $\operatorname{Re}\mu=0$ and $\mu\neq 0$. In the latter event we can easily show that $w_2(z)$ cannot satisfy both (18·1) and (18·2) as $z\to 0$ along a given straight line.

From (14·1) and (17·4) we have

$$\frac{d^2}{dz^2}\{W_1(z)-L_m(z)\}-\frac{1}{z}\frac{d}{dz}\{W_1(z)-L_m(z)\}-\left\{u^2+\frac{\mu^2-1}{z^2}+f(z)\right\}\{W_1(z)-L_m(z)\} = -R_m(z), \tag{18·4}$$

with $\mathscr{Z}=I$ in the definitions of $L_m(z)$ and $R_m(z)$.

Let z_1 be a point of the region $\mathbf{H}_1(\theta,\mu)$, defined in § 14, and suppose that

$$(p-\tfrac{1}{2})\pi\leqslant\arg z_1\leqslant(p+\tfrac{1}{2})\pi, \tag{18·5}$$

where p is an arbitrary integer. Then condition (iv) of § 14 shows that the postulated path $\mathscr{P}$ joining z_1 and the origin must also lie in the sector (18·5). We define the sequence $h_{m,n}(u,\theta,\mu,z_1,\zeta)\equiv h_{m,n}(\zeta)$ by $h_{m,0}(\zeta)=0$ and

$$h_{m,n}(\zeta) = \zeta e^{-p\mu\pi i}\int_0^\zeta\{I_\mu(u\zeta)K_\mu(ut e^{-p\pi i})-K_\mu(u\zeta e^{-p\pi i})I_\mu(ut)\}\{f(t)h_{m,n-1}(t)-R_m(t)\}dt + \alpha_m\zeta I_\mu(u\zeta) \quad (n\geqslant 1), \tag{18·6}$$

where the path of integration is the part of $\mathscr{P}$ between ζ and 0, and

$$\alpha_m = A(u,\theta,\mu)-\sum_{s=0}^{m}\frac{A_s(\theta,\mu,0)}{u^s}. \tag{18·7}$$

The branches of the Bessel functions occurring in (18·6) are determined by continuity and the condition $(p-\frac{1}{2})\pi \leqslant \arg\zeta \leqslant (p+\frac{1}{2})\pi$.

Extending the analysis given in Olver 1956, § 12, we may show that

$$|h_{m,n+1}(\zeta)-h_{m,n}(\zeta)| < \left(\frac{k}{u}\right)^n \frac{k_m}{u^{m+1}}|\zeta e^{p\mu\pi i}| V_\mu(u\zeta') \quad (n=0,1,\ldots), \tag{18·8}$$

and
$$|h'_{m,n+1}(\zeta)-h'_{m,n}(\zeta)| < \left(\frac{k}{u}\right)^n \frac{k_m}{u^{m+1}}|e^{p\mu\pi i}|(1+|u\zeta|) V_\mu(u\zeta') \quad (n=0,1,\ldots), \tag{18·9}$$

where V_μ is defined by (16·3), $\zeta' \equiv \zeta e^{-p\pi i}$, and here and elsewhere in this section the generic numbers k and k_m are independent of p as well as of u, θ, μ and z. From these inequalities and the equation

$$h''_{m,n}(\zeta)-\frac{1}{\zeta}h'_{m,n}(\zeta)-\left(u^2+\frac{\mu^2-1}{\zeta^2}\right)h_{m,n}(\zeta) = f(\zeta)\,h_{m,n-1}(\zeta)-R_m(\zeta) \tag{18·10}$$

obtained by differentiation of (18·6), we deduce that

$$|h''_{m,n+1}(\zeta)-h''_{m,n}(\zeta)| < \left(\frac{k}{u}\right)^n \frac{k_m}{u^{m+1}}|e^{p\mu\pi i}|\frac{1+|u\zeta|^2}{|\zeta|} V_\mu(u\zeta'). \tag{18·11}$$

The parametric equation of $\mathscr{P}$ is of the form

$$\zeta = \zeta(\tau) \quad (\tau_0 \leqslant \tau \leqslant \tau_1), \tag{18·12}$$

where τ_0, τ_1 correspond to 0, z_1, respectively. Term-by-term integration of the series

$$S(\tau) = \sum_{n=0}^{\infty}[h''_{m,n+1}\{\zeta(\tau)\}-h''_{m,n}\{\zeta(\tau)\}] \tag{18·13}$$

multiplied by $\zeta'(\tau)$ is valid over the interval (τ, τ_1) if $\tau > \tau_0$ and $u > k$. Hence we obtain

$$\int_\tau^{\tau_1}\left[\int_\tau^{\tau_1} S(\tau)\,\zeta'(\tau)\,d\tau - \sum_{n=0}^{\infty}\{h'_{m,n+1}(z_1)-h'_{m,n}(z_1)\}\right]\zeta'(\tau)\,d\tau + \sum_{n=0}^{\infty}\{h_{m,n+1}(z_1)-h_{m,n}(z_1)\}$$
$$= \sum_{n=0}^{\infty}[h_{m,n+1}\{\zeta(\tau)\}-h_{m,n}\{\zeta(\tau)\}] = T(\tau),$$

say. Differentiation of this result and use of (18·10) enable us to show that $T(\tau)$ is a solution of the transformed form of the differential equation (18·4) which results from the substitution $z = \zeta(\tau)$. Hence we deduce that

$$T(\tau) = W_1^*\{\zeta(\tau)\}-L_m\{\zeta(\tau)\}, \qquad T'(\tau)/\zeta'(\tau) = W_1^{*\prime}\{\zeta(\tau)\}-L'_m\{\zeta(\tau)\}, \tag{18·14}$$

when $\tau_0 < \tau \leqslant \tau_1$, where $W_1^*(z)$ is the solution of (14·1) satisfying the conditions

$$W_1^*(z_1) = L_m(z_1)+T(\tau_1), \qquad W_1^{*\prime}(z_1) = L'_m(z_1)+\{T'(\tau_1)/\zeta'(\tau_1)\}. \tag{18·15}$$

We now examine the limiting form of $T(\tau)$ as $\tau \to \tau_0$. The part of $\mathscr{P}$ lying inside the circle $|z| \leqslant b$ may be deformed into a straight line without infringing conditions (i) to (v) of § 14. Accordingly, for all $\tau_0 < \tau < \tau(\epsilon)$, assignable, we have

$$\int_0^{\zeta(\tau)} \frac{|dt|}{1+|t|^{1+\sigma_1}} < \epsilon, \tag{18·16}$$

where ϵ is an arbitrary positive number. With this restriction on τ we may repeat the analysis used to establish (18·8) and (18·9) from (18·6), and thence deduce that

$$\left|\sum_{n=0}^{\infty}\{h_{m,n+1}(\zeta)-h_{m,n}(\zeta)\}-\alpha_m\zeta I_\mu(u\zeta)\right|<2\epsilon\frac{k_m}{u^{m+1}}\,|\,\zeta\,\mathrm{e}^{p\mu\pi\mathrm{i}}\,|\,V_\mu(u\zeta')$$

and

$$\left|\sum_{n=0}^{\infty}\{h'_{m,n+1}(\zeta)-h'_{m,n}(\zeta)\}-\alpha_m\{(\mu+1)\,I_\mu(u\zeta)+u\zeta I_{\mu+1}(u\zeta)\}\right|<2\epsilon\frac{k_m}{u^{m+1}}\,|\,\mathrm{e}^{p\mu\pi\mathrm{i}}\,|\,(1+|\,u\zeta\,|)\,V_\mu(u\zeta'),$$

when $u>2k$, (cf. (8·11) and (8·12)). Therefore

$$\lim_{\tau\to\tau_0}\{\zeta(\tau)\}^{-\mu-1}T(\tau)=\frac{(\frac{1}{2}u)^\mu}{\Gamma(1+\mu)}\alpha_m,\qquad \lim_{\tau\to\tau_0}\{\zeta(\tau)\}^{-\mu}\frac{T'(\tau)}{\zeta'(\tau)}=(1+\mu)\frac{(\frac{1}{2}u)^\mu}{\Gamma(1+\mu)}\alpha_m. \quad (18·17)$$

Substituting these equations and the limiting forms of $L_m\{\zeta(\tau)\}$ and $L'_m\{\zeta(\tau)\}$, obtained from (17·2) and (17·3), in (18·14), we see that $W_1^*\{\zeta(\tau)\}$ and $W_1^{*\prime}\{\zeta(\tau)\}$ satisfy the conditions (18·1) and (18·2) with z replaced by $\zeta(\tau)$. Therefore $W_1^*(z)=W_1(z)$.

If $u>2k$, we derive from (18·8) and (18·9)

$$|\,W_1(z_1)-L_m(z_1)\,|<2k_m u^{-m-1}\,|\,z_1\,\mathrm{e}^{p\mu\pi\mathrm{i}}\,|\,V_\mu(uz'_1), \quad (18·18)$$

and

$$|\,W'_1(z_1)-L'_m(z_1)\,|<2k_m u^{-m-1}\,|\,\mathrm{e}^{p\mu\pi\mathrm{i}}\,|\,(1+|\,uz_1\,|)\,V_\mu(uz'_1), \quad (18·19)$$

where $z'_1\equiv z_1\,\mathrm{e}^{-p\pi\mathrm{i}}$. The error terms can be expressed in the forms given in (14·11) and (14·12) by setting

$$z_1\,\mathrm{e}^{p\mu\pi\mathrm{i}}V_\mu(uz'_1)=V_\mu^{(1)}.z_1I_\mu(uz_1)+V_\mu^{(2)}\frac{z_1^2}{u(1+|\,z_1\,|)}I_{\mu+1}(uz_1), \quad (18·20)$$

$$\mathrm{e}^{p\mu\pi\mathrm{i}}(1+|\,uz_1\,|)\,V_\mu(uz'_1)=V_\mu^{(3)}.(1+|\,z_1\,|)\,I_\mu(uz_1)+V_\mu^{(4)}.uz_1I_{\mu+1}(uz_1), \quad (18·21)$$

where

$$\left.\begin{aligned}&V_\mu^{(1)}=V_\mu(uz'_1)\,/I_\mu(uz'_1) && V_\mu^{(2)}=0\quad (|\,uz_1\,|\leqslant 1),\\ &V_\mu^{(1)}=uz'_1K_{\mu+1}(uz'_1)\;V_\mu(uz'_1), && V_\mu^{(2)}=u^2(1+|\,z_1\,|)\,K_\mu(uz'_1)\,V_\mu(uz'_1)\quad (|uz_1|>1),\end{aligned}\right\} \quad (18·22)$$

and

$$V_\mu^{(3)}=\frac{1+|\,uz_1\,|}{1+|\,z_1\,|}V_\mu^{(1)},\qquad V_\mu^{(4)}=\frac{1+|\,uz_1\,|}{1+|\,z_1\,|}\frac{V_\mu^{(2)}}{u^2}. \quad (18·23)$$

Using (16·1), and (16·2) and the inequality

$$\frac{1}{|\,I_\mu(z)\,|}<k\,|\,z^{-\mu}\,|\quad (\mu\in\mathbf{M},\ |\,z\,|\leqslant 1) \quad (18·24)$$

which can readily be established by use of (16·9), we can prove that

$$|\,V_\mu^{(1)}\,|,u^{-2}\,|\,V_\mu^{(2)}\,|,u^{-1}|\,V_\mu^{(3)}|\text{ and }|\,V_\mu^{(4)}\,|$$

are bounded. We see immediately that (14·11) and (14·12) hold if the term $O(1/u^m)$ occurring in the former is replaced by $O(1/u^{m-1})$. The final form is obtained on changing m into $m+1$.

19. Proof for the solution W_2

We suppose first that $a\equiv a(\theta,\mu)$ is a point of $\mathbf{G}(\theta,\mu)$ which is not at infinity.

We write

$$C_s^*(\theta,\mu,z)=\frac{C_s(\theta,\mu,z)}{1+|\,z\,|}, \quad (19·1)$$

and define $A(u,\theta,\mu)$, $B(u,\theta,\mu)$, $C^*(u,\theta,\mu)$ and $D(u,\theta,\mu)$ to be functions with uniform expansions for large positive u typified by

$$A(u,\theta,\mu) \sim \sum_{s=0}^{\infty} \frac{A_s(\theta,\mu,a)}{u^s}. \tag{19·2}$$

Then $W_2(u,\theta,\mu,z) \equiv W_2(z)$ is defined to be the solution of the differential equation (14·1) satisfying the conditions

$$\left.\begin{aligned} W_2(a) &= aK_\mu(ua)\,A(u,\theta,\mu) - (a/u)\,K_{\mu+1}(ua)\,B(u,\theta,\mu),\\ W_2'(a) &= (1+|a|)\,K_\mu(ua)\,C^*(u,\theta,\mu) - uaK_{\mu+1}(ua)\,D(u,\theta,\mu).\end{aligned}\right\} \tag{19·3}$$

Let z_1 be a point of the region $\mathbf{H}_2(\theta,\mu)$, and let ζ be a typical point of the path $\mathscr{P}$ joining z_1 and a. In the expressions for L_m and R_m given in § 17 we take $\mathscr{Z}_\mu = K_\mu$, $\mathscr{Z}_{\mu+1} = -K_{\mu+1}$. The sequence $h_{m,n}(\zeta)$ is then defined by $h_{m,0}(\zeta) = 0$ and

$$h_{m,n}(\zeta) = \zeta\int^a \{K_\mu(u\zeta)\,I_\mu(ut) - I_\mu(u\zeta)\,K_\mu(ut)\}\{f(t)\,h_{m,n-1}(t) - R_m(t)\}\,dt$$
$$+\alpha_m\zeta I_\mu(u\zeta) + \beta_m\zeta K_\mu(u\zeta) \quad (n\geqslant 1), \tag{19·4}$$

where the path of integration is the part of $\mathscr{P}$ between ζ and a, and α_m, β_m are constants. The branches of the modified Bessel functions have their principal values at a, and elsewhere are determined by continuity.

The constants α_m and β_m are determined by the conditions

$$h_{m,n}(a) = W_2(a) - L_m(a), \qquad h'_{m,n}(a) = W_2'(a) - L_m'(a), \tag{19·5}$$

when $n \geqslant 1$. This implies that

$$\left.\begin{aligned} \alpha_m &= \{uK_{\mu+1}(ua) - (\mu+1)\,a^{-1}K_\mu(ua)\}\{W_2(a) - L_m(a)\} + K_\mu(ua)\,\{W_2'(a) - L_m'(a)\},\\ \beta_m &= \{uI_{\mu+1}(ua) + (\mu+1)\,a^{-1}I_\mu(ua)\}\{W_2(a) - L_m(a)\} - I_\mu(ua)\,\{W_2'(a) - L_m'(a)\}.\end{aligned}\right\} \tag{19·6}$$

Substituting by means of the inequalities (16·1) and (16·2), and using the fact that $|a| \geqslant b$, we can show that

$$|\alpha_m| < k_m u^{-m-1}\,|ua|\,\{X_\mu(ua)\}^2, \qquad |\beta_m| < k_m u^{-m-1}. \tag{19·7}$$

Using (19·7) and extending the analysis given in Olver 1956, § 13, we may show that

$$|h_{m,n+1}(\zeta) - h_{m,n}(\zeta)| < \left(\frac{k}{u}\right)^n \frac{k_m}{u^{m+1}}\,|\zeta|\,X_\mu(u\zeta), \tag{19·8}$$

$$|h'_{m,n+1}(\zeta) - h'_{m,n}(\zeta)| < \left(\frac{k}{u}\right)^n \frac{k_m}{u^{m+1}}\{X_\mu(u\zeta) + |u\zeta|\,X_{\mu+1}(u\zeta)\}, \tag{19·9}$$

and hence that
$$|h''_{m,n+1}(\zeta) - h''_{m,n}(\zeta)| < \left(\frac{k}{u}\right)^n \frac{k_m}{u^{m+1}}\,\frac{1+|u\zeta|^2}{|\zeta|}\,X_\mu(u\zeta). \tag{19·10}$$

These are the key results. Using them we deduce that

$$W_2\{\zeta(\tau)\} - L_m\{\zeta(\tau)\} = T(\tau) \equiv \sum_{n=0}^{\infty} [h_{m,n+1}\{\zeta(\tau)\} - h_{m,n}\{\zeta(\tau)\}], \tag{19·11}$$

$$W_2'\{\zeta(\tau)\} - L_m'\{\zeta(\tau)\} = T'(\tau)/\zeta'(\tau) = \sum_{n=0}^{\infty} [h'_{m,n+1}\zeta(\tau) - h'_{m,n}\{\zeta(\tau)\}], \tag{19·12}$$

when $u > k$, where
$$\zeta = \zeta(\tau) \quad (\tau_0 \leqslant \tau \leqslant \tau_1) \tag{19·13}$$

is the parametric equation of $\mathscr{P}$.

Thus if $u > 2k$ we have

$$|W_2(z_1) - L_m(z_1)| < 2k_m u^{-m-1} |z_1| X_\mu(uz_1), \tag{19·14}$$

$$|W_2'(z_1) - L_m'(z_1)| < 2k_m u^{-m-1}(1 + |uz_1|) X_\mu(uz_1). \tag{19·15}$$

To express the error terms in the form given by equations (14·13) and (14·14), we set

$$z_1 X_\mu(uz_1) = X_\mu^{(1)} . z_1 K_\mu(uz_1) + X_\mu^{(2)} \frac{z_1^2}{u(1+|z_1|)} K_{\mu+1}(uz_1), \tag{19·16}$$

$$(1+|uz_1|) X_\mu(uz_1) = X_\mu^{(3)} . (1+|z_1|) K_\mu(uz_1) + X_\mu^{(4)} . uz_1 K_{\mu+1}(uz_1), \tag{19·17}$$

where $X_\mu^{(1)}$, $X_\mu^{(2)}$, $X_\mu^{(3)}$ and $X_\mu^{(4)}$ are defined as follows. Suppose first $|\mu| \geqslant \delta$, where δ is the arbitrary positive number associated with the definition of $l_\mu(z)$ (cf. (16·4)). Then

$$\left.\begin{aligned} X_\mu^{(1)} &= uz_1 I_{\mu+1}(uz_1) X_\mu(uz_1), \\ X_\mu^{(2)} &= u^2(1+|z_1|) I_\mu(uz_1) X_\mu(uz_1), \end{aligned}\right\} \quad (|uz_1| \leqslant 1), \tag{19·18}$$

$$\left.\begin{aligned} X_\mu^{(1)} &= \pm \frac{uz_1}{\pi i} K_{\mu+1}(uz_1 e^{\mp\pi i}) X_\mu(uz_1), \\ X_\mu^{(2)} &= \pm \frac{u^2(1+|z_1|)}{\pi i} K_\mu(uz_1 e^{\mp\pi i}) X_\mu(uz_1), \end{aligned}\right\} \quad (|uz_1| > 1), \tag{19·19}$$

and

$$X_\mu^{(3)} = \frac{1+|uz_1|}{1+|z_1|} X_\mu^{(1)}, \qquad X_\mu^{(4)} = \frac{1+|uz_1|}{1+|z_1|} \frac{X_\mu^{(2)}}{u^2}. \tag{19·20}$$

The upper signs are taken in (19·19) if $0 \leqslant \arg z_1 < \frac{3}{2}\pi$ and the lower if $-\frac{3}{2}\pi < \arg z_1 < 0$.

Alternatively suppose that $\mu = 0$. Then we define $X_\mu^{(1)}$ and $X_\mu^{(2)}$ by equations (13·25) and (13·26) of Olver 1956, replacing $X_\mu^{(2)}$ by $u^{-2}X_\mu^{(2)}$, and $X_\mu^{(3)}$, $X_\mu^{(4)}$ by equations (19·20) above.

Using (16·1) and (16·2) we can prove that $|X_\mu^{(1)}|$, $u^{-2}|X_\mu^{(2)}|$, $u^{-1}|X_\mu^{(3)}|$ and $|X_\mu^{(4)}|$ are bounded. Changing m into $m+1$ we complete the proof of (14·13) and (14·14) when a is not at infinity.

When a is at infinity modifications have to be made to the proof which are similar to those given in § 8 for case A. The solution $W_2(z)$ is determined by the conditions

$$\left.\begin{aligned} \lim \{(2u/\pi z)^{\frac{1}{2}} e^{uz} W_2(z)\} &= A(u, \theta, \mu) - u^{-1} B(u, \theta, \mu), \\ \lim \{(2u/\pi z)^{\frac{1}{2}} e^{uz} W_2'(z)\} &= -uA(u, \theta, \mu) + B(u, \theta, \mu), \end{aligned}\right\} \tag{19·21}$$

as $z \to a$ along the straight line $\mathscr{L}$. The sequence $h_{m,n}(\zeta)$ is defined by (19·4) with $\alpha_m = 0$ and

$$\beta_m = A(u, \theta, \mu) - \sum_{s=0}^{m} \frac{A_s(\theta, \mu, a)}{u^s} - \frac{1}{u}\left\{B(u, \theta, \mu) - \sum_{s=0}^{m-1} \frac{B_s(\theta, \mu, a)}{u^s}\right\}. \tag{19·22}$$

Other modifications are straightforward. Equations (19·14) and (19·15) again hold and the remainder of the proof is unchanged.

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